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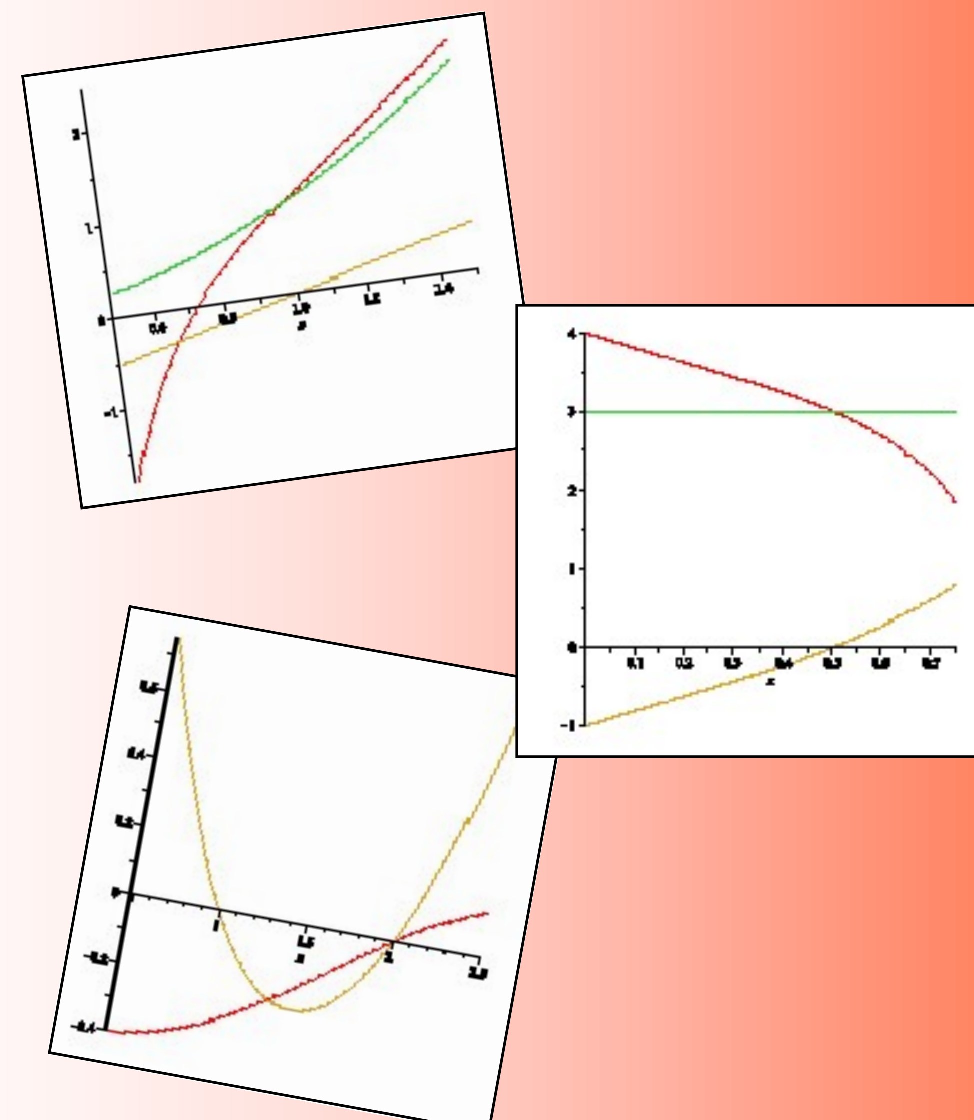
"In most sciences one generation tears down what another has built and what one has established another undoes. In mathematics alone each generations adds a new story to the old structure."
~Hermann Hankel

Utah Mathematics Teacher
Fall/Winter 2010-2011
Volume 3, Issue 1



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Call for Articles

The ***Utah Mathematics Teacher*** seeks articles on issues of interest to mathematics educators, especially K-12 classroom teachers in Utah. All are encouraged to contribute articles and opinions for any section of the journal. Some of the features are: UCTM Leader Spotlight; Letter from the NCTM President; Letter from the UCTM President; Voices from the Classroom; Mathematics for English Language Learners; Puzzle Corner; Recommended Readings and Resources; the Utah Elementary Mathematics Curriculum and Implementation; College and University Research; and others.

Teachers are especially encouraged to submit articles for ***Voices From the Classroom***, including inspirational stories, exemplary lessons, beginning teacher ideas; or managements tools. Sample ideas are (but not limited to) focused on the CMI framework, teachers or districts who have successfully implemented the CMI, Inquiry based calculus, new math programs K-12, the U of U/ Jordan District partnership master's degree program, Cross-district Algebra assessments, and many others.

Manuscripts, including tables and figures, should be typed in Microsoft Word and submitted electronically as an e-mail attachment to Christine Walker (Christine.Walker@uvu.edu). A cover letter containing author's name, address, affiliations, phone, e-mail address and the article's intended audience should be included. Items for ***Beehive Math News*** include, but are not limited to, NCTM affiliated group announcements, advertisements of upcoming professional meetings, and member updates.

UCTM Awards '10-11

Karl Jones Award

Kim Bahr

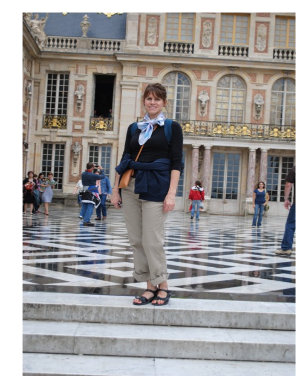
Kim's favorite things about teaching mathematics include discovering how much children know and how deeply they can think about mathematical ideas. She has learned to stop and listen before stopping the students. Kim said, "It is amazing how much I learn from students. I get excited when they come up with solutions on their own. I enjoy discussing and discovering new ideas with them. Teaching is a joy to me and teaching mathematics in an inquiry-based setting is the icing on the cake!"

Don Clark Award

Ray Barton

Ray has earned the highest respect of his colleagues, students, and parents as an exceptional teacher. He fosters respect and admiration by his ability to motivate students to excel to their full potential. He incorporates a variety of instructional materials, teaching strategies, and technology with an energy that engages students yet encourages student responsibility. His classroom structure inspires students to make their own discoveries and draw their own conclusions. His interactive and hand-on style ensures student success. Ray takes great pride and satisfaction, ensuring that each student is actively engaged in the work they produce and promotes mutual respect through example.

George Shell Award



Hilary Moore

Hilary is thrilled to be a part of the teaching profession. She has a background in art history, economics and mathematics and prior to teaching served as a VISTA volunteer. She has worked as a stockbroker and trader, and sold fine art. She is now in her ninth year of teaching and can truly say that she thoroughly enjoys the trials, challenges and successes of being a classroom teacher. She enjoys developing curriculum and continues to seek out new and creative ways to get students engaged in mathematics. Hilary has always considered herself a lifelong learner and teaching provides so many opportunities to learn, grow and change as a person and a professional.

Muffet Reeves Award

Gayle Cloke

Gayle began teaching workshops offered by USOE and the Davis School District in the summer of 1985. In 1996 she was hired as a full-time Math Specialist in the Davis School District and continued to do contract work with the state and other school districts. Gayle has presented at and assisted in organizing many local, state, and national conferences, serving on the UCTM Board of directors for several years.

In 2003, Gayle retired and left Utah to be closer to her daughters. She was a Math Consultant for McREL for a year, and taught in two Colorado schools. After returning home Gayle coordinated a five-district Math Improvement Project for Grades 4-6. She currently works as a Math Coach in the Weber School District.

Gayle had the pleasure of working with and learning from Muffet Reeves. Quoting Gayle…"She was an amazing teacher, a master of mathematics, a motivator of teachers, and a gracious friend. I am honored to receive an award in honor of Muffet."

Presidential Award
Secondary Awarded

Janet Sutorius

Janet Sutorius (23 years teaching) is a graduate of Brigham Young University where she earned a bachelor's and master's degree in math education. She teaches fulltime at Juab High School, while also serving as the district's math specialist. Recently she has worked to help develop the state professional development workshops, *Focus on Functions* and *All Things Rational*. Janet has been an instructor for math endorsement classes for Southern Utah University and Snow College, yet also enjoys reviewing transcripts for NCTM. She has made it possible for teachers in rural Utah to participate in master's degree programs (education with an emphasis in elementary mathematics). Last summer, 25 teachers graduated with their masters degrees as a result of Janet's efforts. How cool is that! As the recipient of this award Janet will visit Washington D.C. and receive a $10,000 honorarium.

Presidential Award
Elementary State Finalists

Cathrine Ermer

Cathrine was born and raised in a small town in northern New Mexico called Questa. In 1986, she graduated from the University of New Mexico with a Master's Degree in Elementary Education and a minor in Early Childhood Studies. She has taught 4th, 2nd, and 1st grade (currently at Adams Elementary in Logan City School district) during her 12 year teaching career. She believes that every student is capable of academic success and knows that her continued professional growth is critical to high student achievement, fortunate enough to work in a district that is committed to effective instruction. She has been married for 24 years to a retired Air Force officer. They have a daughter and three sons.

Linda L'Ai

Linda L'Ai is currently a fourth grade teacher at the Edith Bowen Laboratory School. She graduated with a B.S. in Biological Sciences from the University of California, Davis, also receiving her Elementary and Secondary Teaching Credentials. She received her Master's Degree and Bilingual Certification (Spanish) from California State University, Sacramento. Her major interest is in the integration of math and science in all areas of the curriculum and providing students with learning experiences in a meaningful context. This past summer she worked with math teachers in the Mayan highlands of Guatemala.

Melanie Robbins

Melanie has been teaching in the Salt Lake District for 16 years, eight of those years in a Title 1 school, benefitting from PDE and great mentors. She has taught: K, 1st, 2nd, 3rd, and 4th grades. By special invitation she joined a group of teachers/coaches that began looking deeply at mathematical understanding. Her classroom transformed to a lab classroom where teachers could come and observe various approaches to curriculum, instruction, and assessment. In 2007-2008 she received a sabbatical to study with Catherine Twomey Fosnot (Professor of Education at the City College of New York and Director of Mathematics in the City, a national center for professional development). Melanie now teaches 4th grade at Uintah Elementary and has a passion for helping children mathematize their world!

Utah Mathematics Teacher

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NOTES:

Grades 9-12

Discovering Derivatives

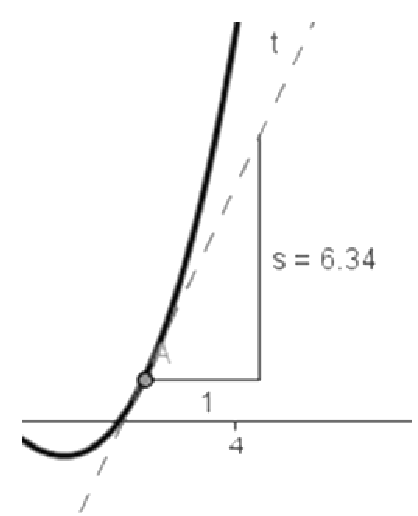
Jan Jenkins, Pre-Service Student, Utah Valley University

As a middle-age student returning to college, I was amazed at the utilization and implementation of technology introduced to facilitate student understanding of mathematical concepts. During my generation buzz words such as “discovery or collaborate learning” had not yet been introduced. The only collaborating or discovery involved students “collaborating” on the best techniques for memorization and “discovering” what the teacher really meant.

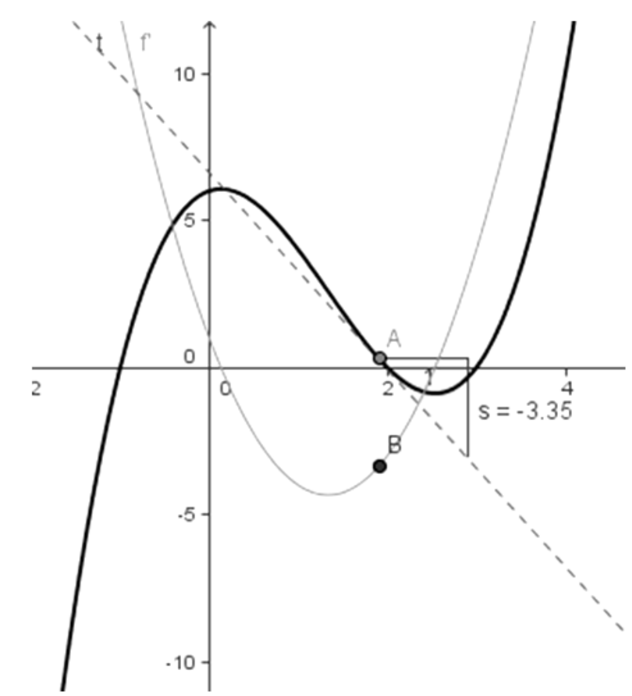
As part of the curriculum to certify as a level 4 mathematics teacher at Utah Valley University we are required to take a course that introduces technology into the mathematics classroom. The technology course introduced me to the concept of “discovery learning” through a program called *Geogebra*. During one of the assignments, we were required to create an interactive worksheet that would graph a function and a line tangent to the function. Thus being interactive, we would be able to move the point along the graph of the function, thereby changing the tangent line. We then attached a “tracer” to the slope so as the slope of the tangent line changed, the tracer was plotting the slope of the tangent line at that point (see graph “a” below). As the tangent line moved through the function, the tracer plotted out the slope for each point (the derivative of the function…see graph “b” below).

Wow! What an incredible visualization of a derivative. The interactive visualization of the relationship between slope of the tangent line and the derivative of the function can facilitate deeper understanding of what a derivative really means. Consider…a “picture is worth a thousand words,” an interactive picture is priceless, as it relates to student understanding!

Graph a



Graph b



General Interest

ESL Learners in the Mathematics Classroom

_________Matthew Rhees, Pre-Service Student, Utah Valley University, dual major Spanish/Mathematics

Have you ever wondered what it would be like to learn a new language? What about learning a language within another language? The very idea can be a tricky one to comprehend but that is exactly what ESL learner's are attempting to do within your mathematics classrooms. Math can be thought of as a language, or at the very least, the terminology that mathematics instructors use can be quite challenging even to those who fully comprehend most aspects of the English language. In more simple cases such as "two times three" the word <u>times</u> could be viewed from an ESL learner as a word dealing with time of day, which could be quite confusing if placed between 2 and 3. They would be left wondering what that would mean. The reason we usually do not see how these types of language barriers affect the way ESL students learn is because we assume that these terms are universal and need no explanation.

Imagine you were first learning Spanish and you were given this problem. "Juan va a comprar na camisa verde, 18 dolares; zapatos negros, 50 dolares; y abrigo, 150 dolares. El impuesto es el 8 porciento. Cuanto paga Juan? " With limited understanding in Spanish this is understood to be Juan purchasing a shirt, shoes, and a coat with various amounts. Yet, built within the problem there is a tax amount and a reference to the total amount paid; difficult for many beginners.

- The question remains…how do we teach ESL student's mathematical concepts? A few pointers can help.
- Be explicit in teaching mathematics vocabulary.
- Provide opportunities for students to listen to other students explain their strategies and mathematical thinking.
- Allow students to verbalize their mathematical thinking one-on-one with other students or instructor, rather than in front of a larger group.
- Create a learning environment that accepts or invites student questioning.

If ESL students can be given opportunities to listen (to both instructors and peers), ask questions of both groups and practice the terminology, they can then focus on the mathematics and less on terminology. The key to understanding mathematics for an ESL student is recognition by the instructor the mathematical terms or words that are not universally known to both groups as meaning the same thing, followed by an explicit explanation of what the terms are interpreted to mean.

Utah's Largest Math Event

Utah's Largest Math Event (ULME) provides an opportunity for Utah teachers and students to join together in meaningful mathematics. The ULME activities are designed to encourage students to work together creatively as they solve rich and challenging problems.

Teachers conduct the event's activities in their classroom, and submit up to 3 samples of student work to compete for prizes. The winning students (and their teachers) in each grade level receive a monetary award. Those students whose work is considered for competition at the state level receive certificates of Honorable Mention, and each student in the class receives a certificate of Participation.

Presidents Message

____Christine Walker

It is my privilege to introduce the third volume of the *Utah Mathematics Teacher*. Launched in 2008 as a general interest public school teacher journal to showcase the talent of our very own teachers, the journal reflects the depth and diversity of thought and academic excellence that Utah teachers strive to achieve.

The journal's publication is the culmination of a six-month editorial process, carried out by the UCTM board and the dedicated reviewers. I would like to take this opportunity to formally thank each of them for their committed service in this ongoing work, and formally apologize for any unnoticed mistakes.

The fall 2010 theme, *Teaching the Next Generation: meeting the needs of all students*, focuses on a core definition of "equity." Reiterating a response received from a member, "Equality refers to the idea of everyone getting equal access to the door of the building that represents education, while equity refers to being able to exit the building with "high comparable outcomes." That will not mean the same treatment and resources. 'Serving all students all of the time' will require different treatment and resources.

One of the implications of the equity definition above is where the responsibility lies for getting students to "high comparable outcomes." If the educational system provides equal input, the responsibility for students to catch up to being on grade level lies with the students. However, if the responsibility is on the system to do whatever it takes to achieve "high comparable outcomes," then we in the system bear responsibility, as well as the students. We are then dealing not with an achievement gap but an instructional gap" (Lee Vanhille, a math coach in the Salt Lake School District).

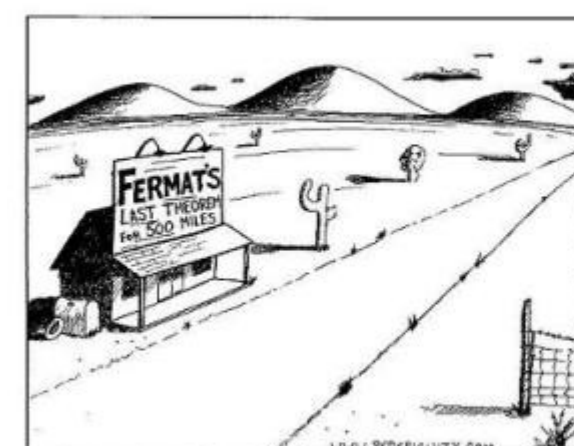
These are the issues you face each day. It is a life-time struggle with the goal that within these pages of the 3rd Volume you will find some ideas that will inspire you to continue in this great work that each of you do.

It was a joy to serve as your president and I look forward to continuous service as editor of the *Utah Mathematics Teacher* as I turn the rein of presidency over to Logan Toone. It has been a wonderful year of working with and alongside many of you as you go about your task of educating the next generation of Utah leaders. Thank you each, for your dedicated service to the children in our great state.

Sincerely,

Christine Walker

Dr. Christine Walker, UCTM President



NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

NCTM Mobilizes CCSS Implementation Support and Continues Its Focus on Reasoning and Sense Making

by NCTM President J. Michael Shaughnessy
NCTM Summing Up, September 2010

I wish you all the best as we start this new school year and greet our new crop of students. As teachers, teacher educators, supervisors, publishers, parents, and administrators, let's not forget that though we are working in challenging times, teaching mathematics to kids is still the absolute best job in the world. We are challenged, but we are also blessed. We are performing vital work in a vibrant profession. In this message, as evidence of that fact, I share updates on two areas of NCTM's support for the implementation of the Common Core State Standards, and on the Council's ongoing initiative to promote reasoning and sense making in mathematics.

NCTM Mobilizes CCSS Interpretation and Implementation Support

The Common Core State Standards (CCSS) for mathematics and language arts were released this past June. Since then, 37 states have decided to adopt and implement the new standards over the next several years. Some states are planning to phase them in over a period of years, while others plan to implement large portions of them at one time. The interpretation and implementation of the CCSS pose many challenges for all stakeholders in the mathematics education of our students. Teachers, teacher leaders, supervisors, state leaders of curriculum and instruction, teacher educators, researchers in mathematics education, and of course, families and caregivers—all of us—will need to work together to implement the CCSS successfully. To provide support to teachers, schools, districts, and states, NCTM has mobilized three new task forces charged with creating various support materials and recommending some next steps for the Council to take as the CCSS proceeds.

First, members of the NCTM Board of Directors are creating a set of short PowerPoint presentations for use by teachers, teacher leaders, and supervisors to provide needed information about the Common Core Standards. An overview presentation on CCSS is already available to NCTM members and is posted on the NCTM Web site. Subsequent presentations that are specific to grade levels will be posted in the coming weeks alongside this overview presentation about CCSS.

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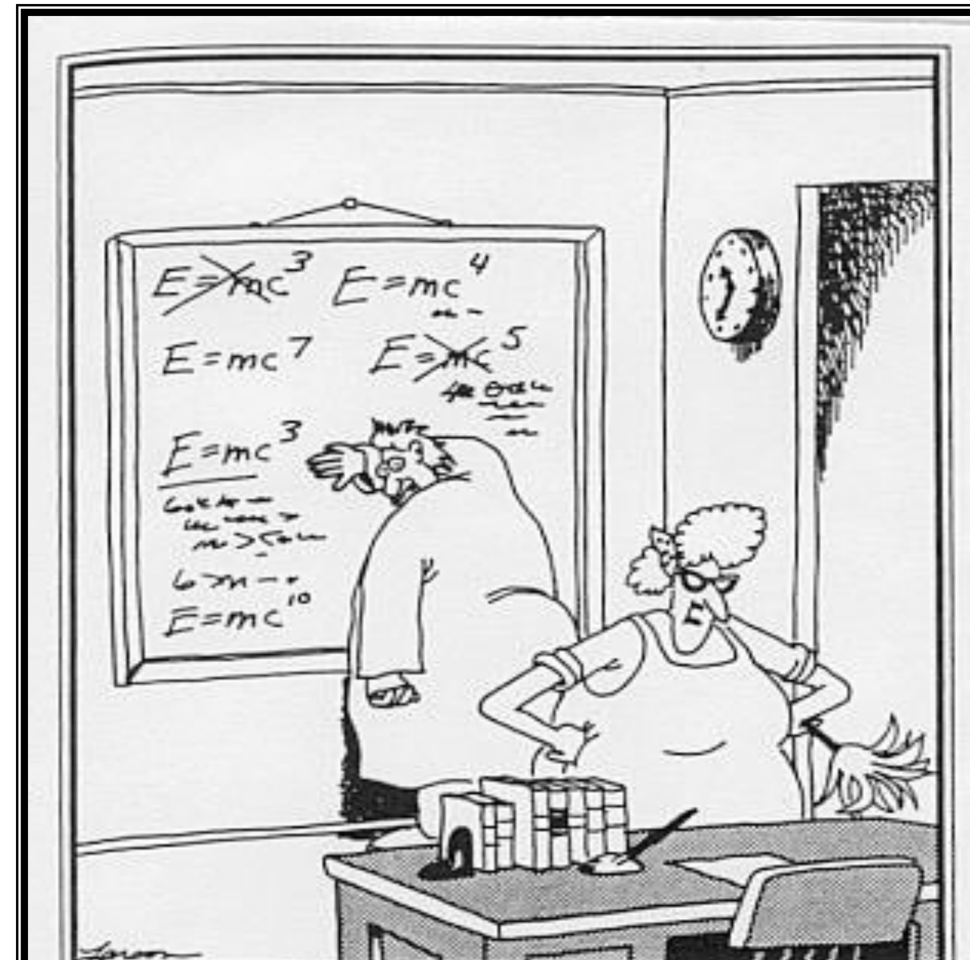

"Now that desk looks better. Everything's squared away, yessir, squaaaaaared away."

Factor 3 (Pedagogical Knowledge)

Item	Factor Loadings
6. Modify instruction, practice, dialog, and assessment for learners who require special education accommodations.	.633
7. Modify curriculum to meet the needs of English language learners.	.465
8. Identify and address special learning needs or difficulties.	.756
9. Address the needs of students who receive special education services.	.721

Factor 4 (Curricular Knowledge)

Item	Factor Loadings
12. Use the standards and objects of the Utah State Core Curriculum in selecting curriculum to use for instruction.	.845
13. Use the state's Core Curriculum and performance standards to plan instruction.	.842
17. Use standardized mathematics assessments to guide your decision about what skills, concepts, and processes to teach.	.604

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Second, an NCTM task force has been drafting a document that connects NCTM's major curriculum and standards publications to the CCSS. NCTM's previous work—especially in *Principles and Standards for School Mathematics, Curriculum Focal Points,* and *Focus in High School Mathematics,* but in other materials as well—can provide support for teachers, schools, and districts as they implement the CCSS. A working draft, currently titled *A Guide to Interpretation and Implementation of the Common Core Standards,* is undergoing review and will be available to members in electronic form in early fall, also to be posted on the NCTM Web site.

Finally, NCTM has joined three other mathematics education organizations—the Association of Mathematics Teacher Educators (AMTE), the National Council of Supervisors of Mathematics (NCSM), and the Association of State Supervisors of Mathematics (ASSM)—to form a task force charged with formulating recommendations for ongoing CCSS related action by our organizations. The report from this joint task force, along with their recommendations, will be sent to the Board of Directors of the four organizations in the next several weeks.

As you can see, the Council has been very proactive this summer in launching new work to support our members as the Common Core Standards are implemented. Keep an eye on the NCTM Web site, and this newsletter, as more CCSS support resources appear.

NCTM Continues to Promote Its Focus on Reasoning and Sense Making in Mathematics

In my first President's Message in the May issue of *Summing Up,* I shared information about NCTM's ongoing initiative to implement the vision of mathematics teaching and learning as laid out in *Focus in High School Mathematics: Reasoning and Sense Making.* In mid September, NCTM will publish the third book in the Focus in High School Mathematics series—*Focus in High School Mathematics: Reasoning and Sense Making in Geometry*—to accompany the companion volumes already published on algebra and statistics and probability. Two other books are under development, one ensuring that *all* high school students have opportunities to participate in reasoning and sense making every day in their classrooms, and another on the importance of integrating technology as we promote reasoning and sense making.

These are great new resources from NCTM. However, as we all know, just having good written materials is not enough to change the way that mathematics is taught—particularly at the secondary level. It is not the case that "if you write it, they will come." We have to live, eat, breathe, and sleep reasoning and sense making. We have to provide examples for classroom teachers that show real teachers and students engaged in reasoning and sense making. We have to provide opportunities for communities of teachers to engage in thinking about how they themselves can incorporate more opportunities for students to reason and share their thinking in everyday classroom work. As outlined in my May message, this ongoing effort to promote student reasoning has been progressing on three fronts.

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First, a preliminary set of Reasoning and Sense Making tasks has been under development and will soon be posted on the NCTM Web site for download and use by NCTM members. Accompanying the tasks are suggestions for classroom implementation and discussion, including ideas about how and where to elicit student thinking and sharing, possible follow-up assessment activities, and how the task connects to both the *Mathematical Practices* in the Common Core State Standards and the NCTM Process Standards. NCTM plans to expand this task bank over the next several years and to include samples of students' work, responses, and reasoning along with the tasks. In addition to a bank of reasoning and sense making tasks, NCTM has developed a PowerPoint presentation on reasoning and sense making and Presenter Guidelines that are posted on the NCTM Web site. This presentation is available for download by all NCTM members. It is intended to provide teachers, teacher educators, and supervisors with information to share with varied audiences. The presentation includes both general information about the reasoning and sense making initiative, as well as some specific examples of tasks that can be used by members as they talk with other teachers and administrators, and families and the business community.

Second, our work on the creation and compiling of reasoning and sense making video clips has been proceeding. NCTM plans to have some video examples of secondary students engaged in reasoning and sense making available early next year. The video clips will be accompanied by clusters of support materials, including discussion and reflection materials as well as student work.

Third, plans continue for a special conference next summer on "Reasoning and Sense Making in Secondary Classrooms." We anticipate that this conference will be held in early August 2011. Dates and location of this special NCTM conference will be revealed soon, so stay tuned! This conference will provide a unique opportunity for teachers, teacher leaders, teacher educators, and leaders of professional development to gather, to actively participate in work sessions on mathematical reasoning and sense making, to listen to some of NCTM's best motivational speakers on implementation strategies, and to have a discussion about video clips of students engaged in reasoning and sense making.

An initial set of reasoning and sense making resources (including materials addressing a range of audiences) is available at www.nctm.org/hsfocus, and much more will be added in the coming months as our work progresses.

J. Michael Shaughnessy
President, National Council of Teachers of Mathematics
NCTM Summing Up, September 2010

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Appendix A

Results of Exploratory Factor Analysis with Principal Axis Factoring

Factor 1 (Mathematical Knowledge)

Item	Factor Loading
22. Explain the algorithm of "invert and multiply" for dividing fractions to students both pictorially and numerically.	.476
24. Explain simplification rules such as why Ö(x+y)^2=(x+y) but that Ö(x^2+y^2)^1 (x+y) in a manner that is accessible to secondary students.	.772
25. Explain mathematics symbols in a manner that helps students understand their mathematical meaning, i.e., helping students understand the difference between $2x$, x^2 and 2^x.	.755
26. Explain why multiplying two negative numbers renders a positive product.	.510
27. Explain the algorithm for an integral using area.	.709
28. Explain the relationship between area models for multiplication, the standard algorithm for multiplication of multi-digit numbers, and the distributive property.	.709
29. Explain why multiplication involving two fractions renders a product smaller than both factors.	.714
30. Prove the quadratic equation.	.747
31. Explain the difference between polynomial and exponential functions.	.826
32. Explain graphing transformation rules (why does f(x-h)+k the graph of f(x) k vertically and h horizontally).	.753
33. Explain why one would want to convert rectangular coordinates to polar coordinates or polar coordinates to rectangular coordinates.	.673
34. Prove fundamental trigonometric identities ($1+\tan^2 x=\sec^2 x$).	.804

Factor 2 (Pedagogical Content Knowledge)

Item	Factor Loadings
16. Take into account students' prior understandings about mathematics when planning curriculum and instruction.	.597
18. Help students move from concrete understandings of mathematics to abstract understandings i.e., teach student how to draw pictures of problem situations and then use the picture to write a mathematical expression or equation for the problem.	.679
19. Help students use prior mathematical understandings to build new understandings, i.e., help student connect adding simple fractions to adding algebraic fractions.	.577
20. Help students use comprehension strategies in mathematics to understand problems and make predictions.	.556

Table 5. Within-subject effects for four domains of teacher knowledge

Source	*df*	Mean Square	*F*	p
Four Domains				
Sphericity Assumed	3.00	5.75	24.03	.000
Greenhouse-Geisser	2.73	6.33	24.03	.000
Huynh-Feldt	2.81	6.14	24.03	.000
Lower-bound	1.00	17.26	24.03	.000
Error (Four Domains)				
Sphericity Assumed	288.00	.25		
Greenhouse-Geisser	261.67	.26		
Huynh-Feldt	270.06	.26		
Lower-bound	96.00	.72		

Table 6. Correlations coefficients of the domains of teacher knowledge

	Mathematical Knowledge	Pedagogical Content Knowledge	Pedagogical Knowledge	Curricular Knowledge
Mathematical Knowledge	1.0	0.62**	0.46**	0.44**
Pedagogical Content Knowledge		1.0	0.57**	0.60**
Pedagogical Knowledge			1.0	0.54**
Curricular Knowledge				1.0

***p*<0.01

Featured Article

Division Algorithm and Rational Functions

Scott Lewis, Utah Valley University

Division Algorithm and Rational Functions

Scott C. Lewis

The division algorithm may be stated as follows: *If $P(x)$ and $D(x)$ are polynomials with $D(x) \neq 0$, and if the degree of $P(x)$ is greater than or equal to the degree of $D(x)$, then unique polynomials $Q(x)$ and $r(x)$ exist such that*

$$P(x) = D(x)Q(x) + R(x)$$

with the degree of $R(x)$ is less than the degree of $D(x)$. The division algorithm is a most important tool in understanding polynomial functions. From it we get the Factor and Remainder Theorems, and hence the ideas behind factoring polynomials and finding zeros. Most college algebra books include a chapter on graphing and factoring polynomials but only one short section on rational functions. The division algorithm also gives important insights into the less studied rational functions, namely asymptotic behavior.

To explore the implications of the division algorithm in regard to rational functions consider the alternate form of the division algorithm found by dividing through the equation by $D(x)$ to get

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

and then considering $\frac{P(x)}{D(x)}$ as the rational function to be investigated. The right side of the equation, $Q(x) + \frac{R(x)}{D(x)}$, reveals some interesting information about asymptotic end behavior in rational functions. We will consider several cases.

We begin by considering the case where $deg(P(x)) < deg(D(x))$. While this case is not allowed in the formal statement of the division algorithm (since degrees don't match statement), it may still be considered in the alternate form if we assign $Q(x)$ to be identically 0, in which case, $R(x) = P(x)$. As in standard texts [1] or [2] we note that the horizontal asymptote is $y = 0$. Thus $y = Q(x)(= 0)$ is the horizontal asymptote. Even more can be inferred; The graph of $y = \frac{P(x)}{D(x)}$ crosses the horizontal asymptote when $R(x) = 0(= P(x))$. Thus the zeros of $R(x)$ that are not zeros of $D(x)$ give intersections between the asymptote and the graph. Consider the following example.

Example 1:Consider $f(x) = \frac{x^2-3x+2}{x^3-4x^2+8x-5}$. The horizontal asymptote is $y = 0$ and the remainder is $R(x) = x^2 - 3x + 2 = (x+2)(x-1)$. The function $R(x)$ is zero when $x = 1$ and $x = 2$. The function $f(x)$ intersects the asymptote only at $x = 2$ since $D(1) = 0$. Because $x = 1$ is not in the domain of the function no crossing occurs (See figure 1 below).

Next suppose $deg(P(x)) = deg(D(x))$ the horizontal asymptote is $y = \frac{a}{b}$, where a is the leading coefficient of $P(x)$ and b is the leading coefficient of $D(x)$. But again notice that $Q(x) = \frac{a}{b}$. Hence, $y = Q(x)$ is the horizontal asymptote. Similar to the previous case it is the zeros of $R(x)$ (but not of $D(x)$) that give locations where the rational function crosses the horizontal asymptote. To see this

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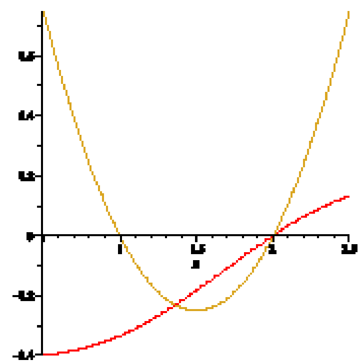

Figure 1: Example 1

note that $\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$, then if $R(x_0) = 0$ we have $\frac{P(x_0)}{D(x_0)} = Q(x_0) = \frac{a}{b}$. Thus the function and the asymptote are at the same level when R(x)=0. The following example illustrates this case.

Example 2:Consider $f(x) = \frac{3x^4+2x^3-x^2+2x-1}{x^4-1} = 3 + \frac{2x^3-x^2+2x-1}{x^4-1}$. The horizontal asymptote is $y = 3$ and the remainder is $R(x) = 2x^3 - x^2 + 2x + 1 = (2x-1)(x^2+1)$. The function f(x) intersects the asymptote only when $2x - 1 = 0$ or at $x = \frac{1}{2}$ (see figure 2 below).

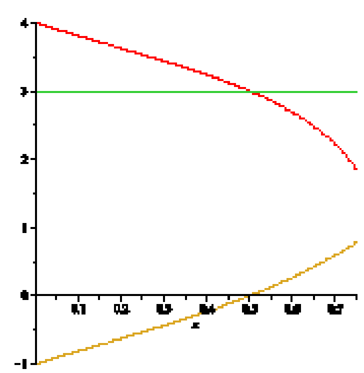

Figure 2: Example 2

Table 2. Descriptive statistics for the domains of teacher knowledge

Domain	Mean	*SD*
Mathematical Knowledge	2.66	.74
Pedagogical Content Knowledge	2.53	.68
Pedagogical Knowledge	2.10	.60
Curricular Knowledge	2.54	.81

Table 3. Cronbach's alpha for the four domains of teacher knowledge

Domain	Cronbach's Alpha
Mathematical Knowledge	.94
Pedagogical Content Knowledge	.86
Pedagogical Knowledge	.81
Curricular Knowledge	.89

Table 4. Pair-wise comparisons of mean differences between four domains of teacher knowledge

		Mean Difference	*SD*	*P*
MK	PCK	.13	.06	.040
	PK	.56	.07	.000
	CK	.12	.08	.151
PCK	MK	-.13	.06	.040
	PK	.43	.06	.000
	CK	-.01	.07	.882
PK	MK	-.56	.07	.000
	PCK	-.43	.06	.000
	CK	-.44	.07	.000
CK	MK	-.12	.08	.151
	PCK	.01	.07	.882
	PK	.44	.07	.000

Table 1. Demographics of Novice Mathematics Teachers

Variable	Response Category	*N*	Percent
Gender			
	Male	35	36.8%
	Female	59	62.1%
	Not Reported	2	2.1%
Licensure Status			
	Licensed	88	92.6%
	Not Licensed	7	7.4%
Years of Mathematics Teaching Experience			
	1st Year	20	21.1%
	2nd Year	19	20%
	3rd Year	22	23.2%
	4th Year	18	18.9%
	5th Year	16	16.8%
Level of Mathematics Endorsement			
	Level II	8	8.4%
	Level III	13	13.7%
	Level IV	63	66.3%
	Does Not Know	3	3.1%
	No Endorsement Yet	8	8.4%
School District/Charter School			
	Alpine	10	10.5%
	Granite	10	10.5%
	Davis	6	6.3%
	Jordan	22	23.2%
	Nebo	6	6.3%
	Tooele	7	7.3%
	Washington	5	5.3%
	Other Public School District	29	30.5%
	Charter School	9	9.5%
Institution from which Licensure was Earned			
	Brigham Young University	23	24.2%
	Southern Utah University	10	10.5%
	University of Phoenix	3	3.1%
	University of Utah	12	12.6%
	Utah State University	13	13.7%
	Utah Valley State (College) University	3	3.1%
	Weber State University	4	4.2%
	Western Governors University	1	1.1%
	Other	15	15.8%
	No Licensure yet	7	7.4%
	Missing	4	4.2%

Finally consider the case when $deg(P(x)) > deg(D(x))$. Here the rational function $\frac{P(x)}{D(x)}$ will have no horizontal asymptote. It will follow some asymptotic behavior given by $y = Q(x)$. For instance, slant (linear) asymptotes occur when $deg(P(x)) = deg(D(x)) + 1$ and are found by dividing $P(x)$ by $D(x)$ and throwing away the remainder term. Similarly, other asymptotic behavior $y = Q(x)$ of higher order polynomials can be found. As with other cases the rational function $\frac{P(x)}{D(x)}$ crosses the asymptote $y = Q(x)$ when $R(x) = 0$ (and $D(x) \neq 0$).

Example 3:Consider the rational function $f(x) = \frac{x^4+x-1}{x} = x^2 + \frac{x-1}{x^2}$. Note that the asymptotic behavior is given by $y = Q(x) = x^2$ and $R(x) = x - 1$.

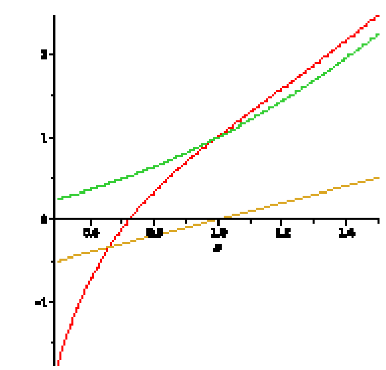

Figure 3: Example 3

The results of this discussion may be summarized as follows:

Theorem *The asymptotic end behavior of a rational function is $y = Q(x)$, where $Q(x)$ is the quotient when dividing $P(x)$ by $D(x)$. Furthermore, the location(s) where $\frac{P(x)}{D(x)}$ cross the asymptotic behavior are the zeros of the remainder $R(x)$ that are not zeros of $D(x)$.*

Using the results of the the theorem above and the Complete Factorization Theorem [1] we also get

Corollary *The number of times a rational function intersects the asymptote y=Q(x) is less than or equal to the degree of the remainder $R(x)$.*

References

[1] J. Stewart, L. Redlin, and S. Watson. *College Algebra, 4th ed.* Thompson Learning, 2004.

[2] C. Young. *College Algebra, 2nd ed.* John Wiley & Sons, 2009.

Grades K-6

Teaching Multiplication Facts With Meaning

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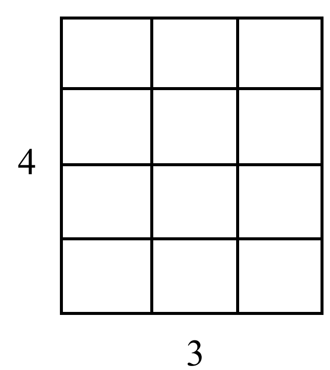
I recently worked with a teacher who stated, “I don’t teach multiplication. My kids just learn their facts.” This made me realize that many teachers don’t know that there is a process through which children learn multiplication and that memorization should be the final step of that process. Even though children are expected to learn multiplication in third and fourth grade, this process also applies to students who didn’t learn the multiplication facts the first time around, regardless of the grade in which they are currently enrolled.

Conceptual Knowledge

The first step to learning multiplication is to build conceptual knowledge. Children at an early age can solve the problem: How many legs do three cats have? A child can draw a picture of three cats and count the numbers of legs, or she can use objects to show three groups of four. A child who is a little older could use an addition number sentence such as 4 + 4 + 4 = 12. All of these strategies help build the meaning of multiplication.

Once the term multiplication is introduced, students will benefit from using models as a basis for conceptual knowledge. As referred to above, a child can form equal-sized groups using objects and count the number in each group by ones or skip count by the number in each group. Arrays are a specific type of model that shows an arrangement of items in rows and columns, such as 4 rows with 3 tiles in each row.



The number line can be used to represent equal-sized jumps. Below is an example of a number line that models 4 x 4. A child makes four equal jumps of 4 and lands on 16. The jumps on the number line can also be correlated to repeated addition.
4 + 4 + 4 + 4 = 16

Once a good understanding of what multiplication means and how it can be represented has been developed, the child is ready to focus on relational thinking.

Figure 1. Current two-domain structure of teacher preparation and licensure

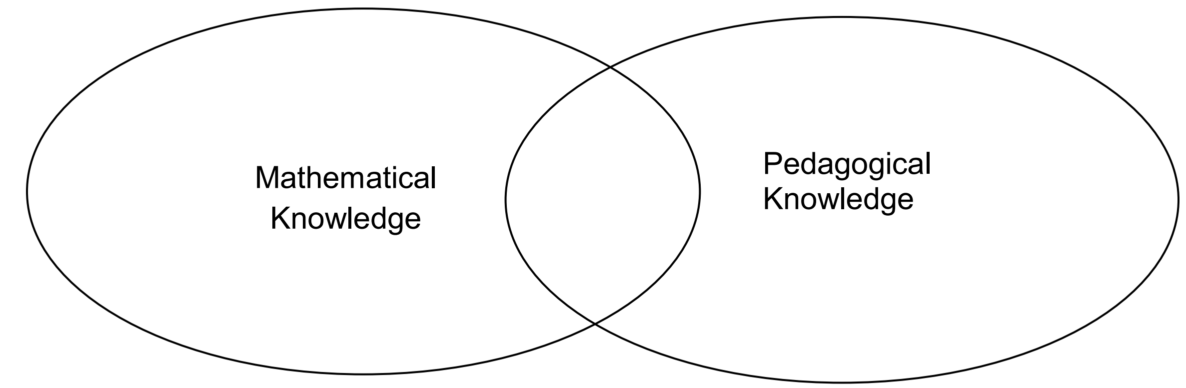
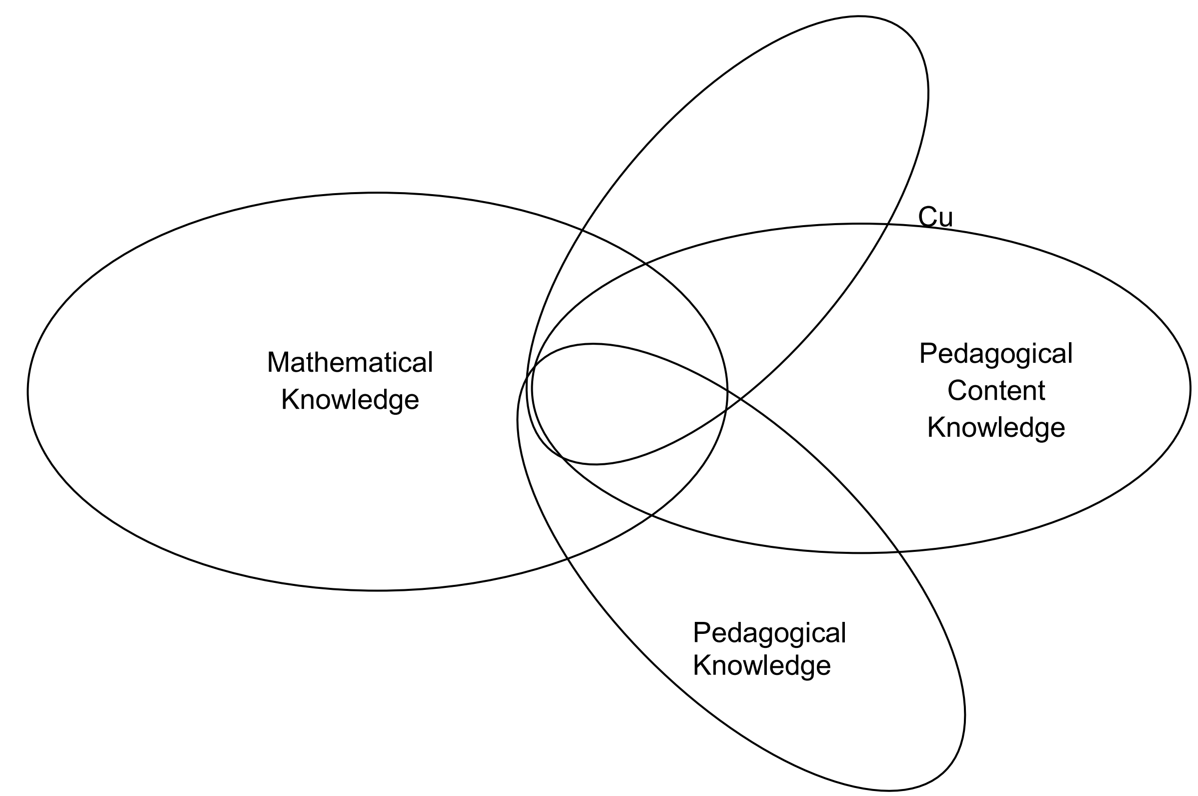


Figure 2. Proposed four-domain structure of teacher preparation and licensure



That the domains are not only distinct and discernable but that novice teachers indicated varied levels of preparedness in the different domains. If the four domains more fully capture the work of novice teachers, as this study's findings suggest, then they offer a new structure for evaluating preparedness to enter the profession of secondary mathematics teaching.

The findings of this study also have implications for institutions of higher education within the state. Individuals within departments of mathematics and colleges of education at Utah's colleges and universities that prepare mathematics teacher candidates need to work more collaboratively to better meet the needs of prospective secondary teachers. Mathematics departments at all institutions currently offer all courses required in the domain of mathematical knowledge (see http://www.schools.utah.gov/cert/Endorsements/docs/endmath.pdf). Within this list of courses, some are specifically designed for individuals preparing to become teachers (e.g. Foundations of Algebra, Euclidian and Non-Euclidian Geometry, and Methods of Teaching Secondary Mathematics), with the rest designed for individuals going into a number of fields and are required of prospective teachers to develop general mathematical knowledge (e.g. College Algebra, Trigonometry, Calculus 1 and 2, Multivariable Calculus, [Calculus based] Probability and Statistics, Linear Algebra, Differential Equations, and Introduction to Analysis). For the courses designed specifically for teacher candidates, mathematics departments should work more closely with departments of teacher education to address issues of pedagogical content knowledge and curricular knowledge. By the same token, departments within colleges of education should work more closely with mathematics departments to address issues of working with students with special education or language needs specifically in mathematics, to develop sheltered instructional strategies specific to mathematics, or understand cultural differences in mathematics algorithmic strategies. Hence, mathematics courses for teachers need to better address pedagogical content and curricular needs of prospective teachers and pedagogy courses need to better address specific strategies in mathematics.

This study aims to help policymakers and educational leaders in secondary mathematics teacher preparation and licensure better understand the effectiveness of Utah's current structure for ensuring that students have quality mathematics teachers. Findings herein suggest that framing preparation and licensure around only two domains, content knowledge and pedagogical knowledge, may not comprehensively address the knowledge and skills needed by secondary mathematics teachers as they begin their work with Utah's student. We must reevaluate the content and structure of coursework in mathematics for individuals going in to secondary teaching. With respect to pedagogical knowledge, survey respondents indicate they feel least prepared in this domain. In terms of the two new proposed domains, pedagogical content knowledge and curricular knowledge, the findings of this study indicate that these are discernable knowledge domains. Not only are these skills discernable and learnable, policy structures must explicitly address these domains in preparation and measure them as part of licensure. For these domains in particular, departments of mathematics and teachers education must work more closely together to better prepare Utah's secondary mathematics teachers. Teacher preparation and licensure must support contemporary schools by preparing and licensing teachers prepared to meet the high quality demands of the role. In order to achieve the desired end of high quality teachers for all students, teacher preparation programs and state licensing bodies must respond to emerging research such as this that provide additional insight into the nature of effective preparation and licensure. The four-domain structure of teacher knowledge represents an improvement in the conceptualization of teacher knowledge by articulating four distinct knowledge components that can be explicitly taught and learned.

Relational Thinking

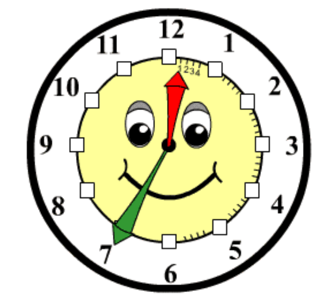
When students develop relational thinking, they are able use what they know to solve facts they don't know. If a child knows the doubles for addition, that knowledge can be used as a strategy for solving the multiplication facts for threes and fours. For example, if he doesn't know the answer to 3 x 8, he can solve 2 x 8 and add on one more group of 8.

$$2 \times 8 = 16;\ 16 + 8 = 24$$

If she doesn't know the answer to 4 x 8, but she knows that 2 x 8 equals 16, she can double 16 for the answer of 32.

$$2 \times 8 = 16;\ 16 + 16 = 32$$

The clock can be used to help solve the five facts. If a child wants to find the answer to 5 x 7, he considers where the minute hand is on the clock when it's on the seven. How many minutes past the hour is it? It is 35 minutes after the hour; therefore, the answer to 5 x 7 is 35.



At this stage, it is important that children are able to recognize and use the commutative property of multiplication, whether they use the term or not. It is not obvious to young learners that 3 x 8 and 8 x 3 both equal 24. Children do not automatically make the connection that 3 sets of 8 objects is equal to 8 sets of 3 objects. Arrays are a good representation when the 3 x 8 array is rotated to show an 8 x 3 array. The answer is the same because nothing was added, and nothing was taken away. When a student recognizes the power of the commutative property, the number of facts she needs to learn is cut nearly in half.

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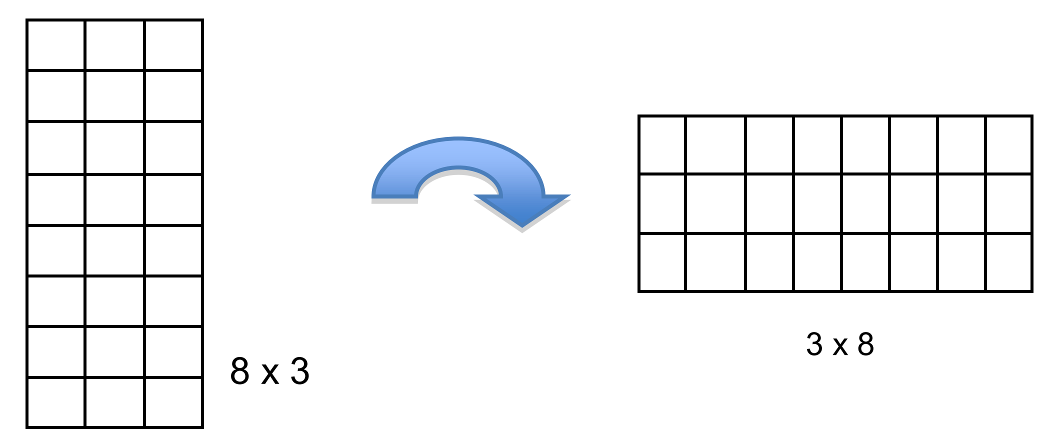
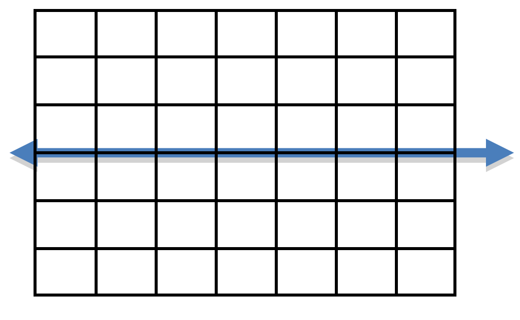


8 x 3

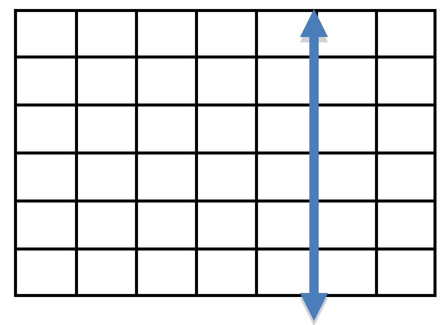
3 x 8

When facts are developed with understanding, children can become skillful at using the facts they know to figure out the facts they don't know using the distributive property. Arrays are also a good way to illustrate this property. When solving for an unknown fact, one factor can be broken up into two parts. Two new multiplication pairs are created, each piece is solved, and then the pieces are joined back together for the answer. For example, 6 x 7 = (3 x 7) + (3 x 7) = 21 + 21 = 42 or it equals (5 x 7) + (1 x 7) = 35 + 7 = 42. In this case the 6 was divided and the 7 was kept whole. Which factor to split and how it is split should be determined by facts the student already knows.

6 x 7 =



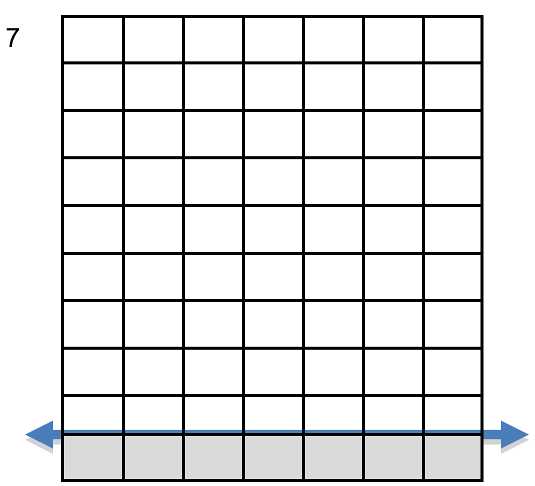
(3 x 7) + (3 x 7) = 21 + 21 = 42



(6 x 5) + (6 x 2) = 30 + 12 = 42

The distributive property is an especially effective strategy to use when learning facts for 9. If a child doesn't know the answer to 9 x 7, he can solve 10 x 7 and take one group of 7 away. For example, 9 x 7 = (10 x 7) - (1 x 7) = 70 - 7 = 63.

9 x 7



(10 x 7) - (1 x 7) = 70 - 6 = 63

Once students have had practice solving multiplication facts using these strategies, they are ready to work toward fluency, or a quick recall of the facts.

Abstract Representation

Understanding the meaning of multiplication and developing strategies to solve unknown facts will put students on the road to fact mastery. You will find that many of your students will have learned many facts through this process. At this point, an assessment becomes necessary to determine which facts the child knows by recall, which he has strategies for, and those with which he is still struggling. For those he is still struggling with, make sure he has conceptual knowledge before helping him develop strategies. For those he has strategies for, more practice is needed. One activity he can do is to create flash cards. On the front of the card he should list the fact to be practiced and a clue that reminds him of the strategy he will use to solve the problem. On the back, the product is written.

closely correlated to both PK, r=.57 p<.01 (2-tailed) and CK, r=.60 p<.01 (2-tailed) than MK to those domains. Thus, MK, the knowledge domain in which novice teachers report feeling most prepared, is most strongly correlated with PCK. PCK, the knowledge domain in which novice teachers report feeling "second most" prepared (along with CK) has a strong correlation to the two other domains, PK and CK. Although means for MK and CK do not statistically differ, these two knowledge domains have a relative lower correlation (r=44, p<.01). The findings of the correlation analysis support the hypothesized overlapping relationship between the four domains of teacher knowledge.

Results from this analysis indicate that novice secondary mathematics teachers in Utah feel that they have different levels of preparedness in each of the domains of teacher knowledge. Utah's novice teachers feel most prepared in the domain of mathematical knowledge and least in the domain of pedagogical knowledge (as was defined in the conceptualization.) Of particular interest is how the levels of preparedness are correlated to each other. Mathematical knowledge was most closely correlated to PCK, supporting the findings of Hill, Ball, and Shilling (2008), and least to CK. But, PCK was more closely correlated to PK and CK. The domains are indeed interrelated. This may indicate that if teacher candidates were better prepared in MK, then their level of preparedness in the other domains may also increase. It is concerning, however, that teachers feel the least prepared to work with students of diverse backgrounds (PK). This results supports the findings of Cochran-Smith, Davis and Fries (2003) that there remains a great need to better prepare teachers to work with students from various backgrounds, efforts in this area render inconsistent results.

Limitations

There are several limitations on this study. First, the low response rate is problematic. Of the 562 novice teachers identified for the survey, 142 teachers responded, but only 95 surveys met all the criteria and could be used. This represents 16.9% of the population. The limited response may raises concerns about the findings and suggests the need to do follow-up investigation. Additionally, this study only assesses the perspective of novice teachers trained and working in Utah. Hence, generalizability is limited to Utah. Further assessment of novice teachers around the country is necessary to make larger generalizations.

Another limitation is the survey instrument used for this research. Since the framework is new, the instrument used to assess the framework is new. The instrument needs further examination and refinement. Many of the items in the survey relating to each of the domains need to be rewritten and refined. Additionally, more items need to be developed for pedagogical knowledge, pedagogical content knowledge, and particularly for curricular knowledge. Items on the survey that did not load on one of the factors representing the four domains of teacher knowledge also need to be carefully assessed to determine if either further exploration of the factors needs to be done, or if the items were simply poor items. Also, the instrument was too long and therefore may have discouraged teachers from participating.

A final limitation is that data for this research relied on teachers' self-reported perceptions of preparedness. Self-reported data can be problematic, particularly in terms of ascertaining where teachers believe they gained their skills and knowledge.

Implications of Findings

The findings of this study have direct implications for preparation and licensure in Utah. The first of these implications is that policy addressing secondary mathematics teacher licensure should be re-evaluated to determine if the domains around which licensure is conceptualized should be expanded. Currently, individuals seeking licensure in secondary mathematics must show competency in mathematical knowledge, as measured by both coursework and passing the Praxis 0061 or 0069 mathematics content exam; and pedagogical knowledge, as measured by both coursework and passing the Praxis PLT exam. The results of this study suggest that explicitly expanding course requirements and assessments into the other two domains of pedagogical content knowledge and curricular knowledge in order to more fully address candidates' readiness to begin the work of teaching. This study's findings indicate

These results indicate that the domains of mathematical knowledge, pedagogical content knowledge, pedagogical knowledge, and curricular knowledge are distinct aspects of the professional knowledge of novice mathematics teachers in Utah. The first interesting insight prompted by the analysis is that respondents distinguish their understanding and ability to explain and do specific mathematical concepts from "helping students" connect to general mathematics. It is important to note that items in the pedagogical content knowledge factor were void of specific mathematical skills whereas those in the mathematical knowledge factor each specifically addressed discrete mathematical skills. This indicates that teachers in the sample feel differently about their ability to "do" and/or "explain" specific mathematical tasks then they feel about their ability to help students with mathematics. This supports Hill, Ball, and Schilling's (2008) finding that some teachers may rely on their knowledge of content and student when evaluating student work, while others may rely on their knowledge of the content making the domains sometimes difficult to distinguish.

Also of note from the analysis is the fact that pedagogical content knowledge, pedagogical knowledge and curricular knowledge each emerged as distinct factors (rather than all together as one factor). This finding directly confirms the proposed four-domain structure of teacher knowledge. Teachers' ability to work with students of diverse backgrounds is distinct from their ability to connect any student to mathematics or their ability to use standardized tools for working with students. As was suggested in the description of the conceptual framework, it is likely that these domains, along with mathematical knowledge, become more interrelated as the teacher becomes more experienced. However, as an individual makes the transition from expert student to novice teachers, the findings support distinguishing among the domains and explicitly addressing each domain separately in teacher preparation and licensure.

In order to determine the extent to which novice mathematics teachers in Utah perceive that they are prepared to do the work of teaching secondary mathematics, a repeated-measures ANOVA was used to compare the overall scores in each of the domains of teacher knowledge. Table 2 above presents the mean of scores for each of the domains. The a simple comparison of the means reveals that novice teachers feel most prepared in the domain of mathematical knowledge, with a mean score of 2.66, and least prepared in the domain of pedagogical knowledge, with a mean score of 2.10. Mean scores for pedagogical content knowledge and curricular knowledge were relatively close at 2.53 and 2.54, respectively, indicating that sample Utah teachers feel fairly equally prepared in each of these two domains. A repeated-measures ANOVA tested for statistically significant differences in the means. Mauchly's test indicated that the assumption of sphericity had been violated (chi-squared=14.19, $p<.05$), therefore the degrees of freedom were corrected using Huynh-Feldt estimates of sphericity (epsilon=.91). The test showed that the differences in mean scores for the domains differed were statistically significant, $F(2.81, 270.06)=24.03$, $p<.001$ (see Table 5). Pair-wise comparisons of mean differences between four domains of teacher knowledge are presented in Table 4. Post hoc pair wise comparisons revealed that although mean MK is significantly higher than both mean PCK and mean PK ($p<.05$ and $p<.001$ respectively), it was not significantly higher than mean CK. Additionally, mean PCK was not significantly different than mean CK though it was significantly higher than mean PK ($p<.05$).

This analysis indicates how teachers in the sample perceive their relative preparedness in each of the domains of teacher knowledge. The findings allow us to better understand in which areas teachers perceive they are better and less well prepared. The teachers in the sample perceive that they are best prepared in MK. Though novice teachers feel more prepared in MK than in PCK, they feel about the same level of preparedness in PCK as they do in CK. It is clear, though, that they feel the least prepared in PK. Their level of preparedness in PK is significantly lower than MK (mean difference of -.56, $p=.000$), CK (mean difference of -.44, $p=.000$), and PCK (mean difference of .43, $p=.000$) respectively.

The four-domain structure of teacher knowledge hypothesizes a degree of correlation between the domains. An examination of correlation coefficients between the domains supports this hypothesis. The correlation coefficients presented in Table 6 clarify that each of the domains is moderately, positively correlated with the others. Mathematical Knowledge is most strongly correlated to PCK, $r=.62$, $p<.01$ (2-tailed) and least with CK, $r=.44$, $p<.01$ (2-tailed). Pedagogical Content Knowledge is more

Front	Back
3 x 8 (2 x 8) + 8	**24**

After a practice period, it is time to reassess the child's knowledge to determine which facts he is now fluent with and those he still needs to practice. There are several tasks too encourage the use of the distributive property. One is a game called "Small Array/Big Array" (Investigations in Number, Data and Space, 2008). In this game, students use two smaller arrays that combine to create a larger array. Another task is to create foldable facts. For this task a child creates two factor pairs by folding the original array along one side. For example, a child folds the 7 x 7 along one side resulting in two pieces, 2 x 7 and 5 x 7 that when solved and combined result in the original product.

Timed Tests

Often timed tests are used as a way to teach the basic facts. Timed tests should not be used as a teaching tool! It is important to remember that unless a student has already developed an efficient strategy to solve an unknown fact, she is being required to get faster using inefficient strategies. Also, for any student who is struggling or having difficulty working under time pressure, a teacher runs the risk of making her hate math or feel that she is just not any good at it. Timed tests may be used as an assessment tool to determine which facts a student knows and which she still needs to practice. But children need opportunities to practice for a minimum of two weeks before they are assessed again. There are a few students who are motivated by timed tests because they perform well under pressure situations. However, since this is not the case with most children, it is best to avoid them altogether.

Conclusion

Knowing multiplication facts is an important mathematical skill. However, it is only useful for a student if it is based in understanding. There are students who can rattle off "their facts" but have no idea how to apply this knowledge in a real-life situation or even what number relationships to which they are referring. For example, 6 x 6 = 6 + 6 + 6 + 6 + 6 + 6 = 6 x 6 = (3 x 6) + (3 x 6) = (5 x 6) + (1 x 6). As teachers, it is our responsibility not only to teach our students the skills they need, but to empower them to apply the information in real-life situations.

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Grades K-8

Area Models and Division of Fractions

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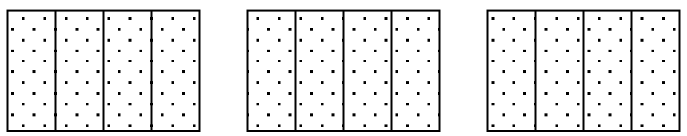
Explanations of the division of fractions tend to focus on the standard algorithm of “invert and multiply.” This overemphasis of manipulation as explanation does a great disservice to students because it does not connect to their intuitive understanding of division – how many of the divisor is in the dividend. Focusing on symbol manipulation also ignores a fundamental model of fractions – the area model. In this article, I will illustrate how we can explain division of fractions by using area models and how these models can help justify the invert-and-multiply algorithm.

Let us start with dividing a whole number by a unit fraction, namely, $3 \div \frac{1}{4}$. Here we want to find how many $\frac{1}{4}$'s are in 3. Figure A illustrates three wholes divided into fourths:

Figure A

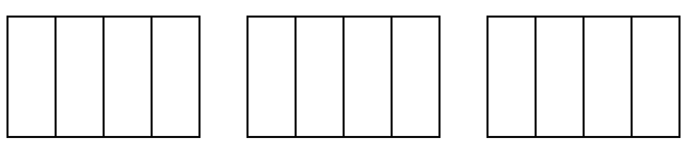


Each small rectangle represents $\frac{1}{4}$ and each larger rectangle has four $\frac{1}{4}$'s for a total of twelve $\frac{1}{4}$'s.

So, $3 \div \frac{1}{4} = 12$.

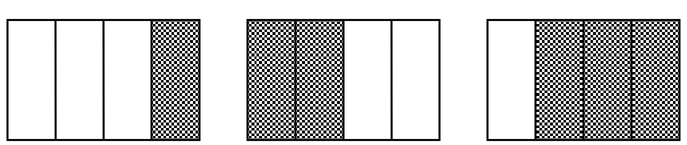
We can increase the difficulty a little by considering $3 \div \frac{3}{4}$. Figure B shows three wholes where each whole has been divided into fourths.

Figure B



Three of these smaller rectangles are $\frac{3}{4}$. In Figure C, the alternate shading differentiates the different $\frac{3}{4}$'s in three wholes.

Figure C



Thirty-five of the respondents were male, 59 were female, and 3 participants did not identify gender. Twenty respondents were in their first year of teaching mathematics, 19 in their second, 22 in their third, 18 in their fourth, and 16 in their fifth year. Eight had a level 2 mathematics endorsement, 13 a level 3 mathematics endorsement, 63 a level 4 mathematics endorsement, 3 did not know what level they had and 8 had not yet received a mathematics endorsement at the time of the survey. Teachers from 22 of the 40 Utah public school districts responded. Additionally, 9 teachers from charter schools responded. Representation from Jordan School District was particularly high, 23.7% of the respondents; this is likely a function of one of the researchers working in Jordan School District and thus being known to study invitees.

The survey instrument was created specifically for the purpose of this study. Items in the first part of the survey were either adapted from the Utah Novice Teachers Research Team (2008) and Darling-Hammond et al.'s study of novice teachers (2002) or were written specifically to address the proposed conceptualization. Each item in Part I of the survey represented a discrete skill or concept involved in the teaching of secondary mathematics. For each item, respondents were asked two questions: “How well prepared are you with respect to the knowledge or skill item listed below?” with a Likert-type scale with four levels: (a) not at all prepared, (b) somewhat prepared, (c) well prepared, and (d) very well prepared; and “Where did you PRIMARILY learn each of the knowledge or skill item listed below?” with seven possible responses: (a) in my college **math** classes, (b) in my college general **education or licensure** classes, (c) in my college **math methods or pedagogy** classes, (d) during my **student teaching** experience, (e) from my own **personal experiences** (e.g., as a student or tutor), (f) during my initial **teaching** experience, and (g) other; please specify. The survey instrument also included items related to the participants' age, gender, race or ethnicity, institution(s) from which the participant earned degrees and licenses, type(s) of degree, highest degree, years of experience, and level of endorsement.

Analysis and Findings

In order to determine if the four-domain structure of teacher knowledge accurately represents the knowledge and skills of novice secondary mathematics teachers in Utah, we conducted an exploratory factor analysis (principal axis factoring) using Varimax rotation with Kaiser Normalization to investigate the validity of the four hypothesized constructs (see Appendix A for a list of items loading on each factor). Using a factor loading cut-off of 0.4, four strong factors corresponding to the four hypothesized domains emerged from the analysis; the four factors accounting for 61.4% of the total variance. The first factor, mathematical knowledge (MK), includes items that relate to doing and understanding mathematics, but not to imparting or teaching that knowledge. The second factor, pedagogical content knowledge (PCK), includes items that focus on the teaching of mathematics to students. The third factor, pedagogical knowledge (PK), includes items that do not reference mathematics specifically but rather the skills necessary to teach all subjects to students of any age. The fourth factor, curricular knowledge (CK) includes items that relate to the state's core curriculum or the use of standardized assessment tools to guide instruction. Scores for each domain of teacher knowledge were created by taking the mean of responses to items that loaded on each respective domain. Table 2 presents the descriptive statistics for each of the four domains of teacher knowledge.

Cronbach's Alpha was used in order to evaluate the internal consistency of each of the domains, For each of the four domains, a > .8,with alphas ranging from .806 to .939; this result indicates strong internal consistency for each construct (see Table 3). The combination of these two tests, the exploratory factor analysis and Cronbach's alpha, offers evidence of both the instrument's reliability and validity for assessing the four domains proposed by the conceptual framework for secondary mathematics teachers' work.

The sets of knowledge and skills that comprise the domains of teacher knowledge outlined above are not completely distinct. In order for a teacher to develop much of the knowledge and skills required to teach math-specific content (pedagogical content knowledge) for example, she must first understand and be able to “do” the mathematics (content knowledge) as well has have knowledge of general pedagogy. Hence, in order for a teacher to be able to help a student understand why one cannot further simplify 2x + 3y, but one can simplify (2x)(3y), the teacher must first know how to accomplish both computations and must understand the underlying principles of addition and multiplication that govern the simplification of each. Once she understands the mathematics, she can then move to articulating what makes the process difficult for a student to understand, the common errors students make with both types of simplification, and how to help a student connect previous knowledge of addition and multiplication with the simplification computations. Although the skills are coupled, they are distinct. The ability to simplify algebraic expressions does not ensure that an individual can teach it. Even in instances where a teacher has a comprehensive knowledge of and skill in mathematical knowledge, pedagogical knowledge, and pedagogical content knowledge, the range of knowledge and skills needed to effectively teach all students is incomplete; curricular knowledge is necessary in order to plan and prepare lessons that meet student knowledge needs as reflected in learning standards. All four components are necessary for quality instruction.

Whereas the current structure of mathematics teacher preparation and licensure presented in Figure 1 emphasizes two domains of teacher knowledge, content knowledge and pedagogical knowledge, the proposed four-domain structure of teacher knowledge in Figure 2 adds the components of pedagogical content knowledge and curricular knowledge. In the four-domain structure, mathematical knowledge is on the left as the largest domain with pedagogical knowledge, pedagogical content knowledge, and curricular knowledge replacing the singular pedagogical knowledge from the two-domain structure. These three domains overlap to indicate two separate ideas: first, that the domains are conceptually related and intertwined, and second, that novice teachers can and do glean some knowledge and skills in one domain as they are focus on gaining knowledge and skill in other domains. The model is also intended to represent the notion of inward movement of the domains; as teachers become more experience and skilled, the domains become more overlapped. As teachers gain experience, their knowledge and skills become more interdependent and integrated.

The study presented here is designed to test the four-domain structure of teacher knowledge by answering the following research questions:

- Does the four-domain structure of teacher knowledge accurately represent the knowledge and skills of novice secondary mathematics teachers in Utah?

To what extent do novice mathematics teachers in Utah perceive they are prepared to do the work of teaching secondary mathematics?

Methods and Data

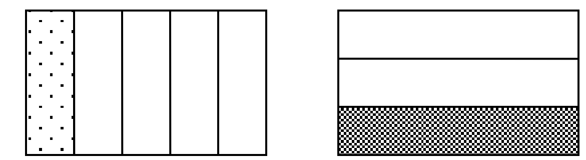
Data for this study were collected via an electronic survey instrument developed specifically for the purposes of this research. A query conducted through the Utah State Office of Education using the 2008-2009 CACTUS database identified teachers who met the desired sample criteria specifically, secondary mathematics teachers (those a teaching assignment in secondary mathematics excluding Math Grade 6 and any Special Education Mathematics codes) with less than 5 years experience. These teachers were invited to participate in the study by completing the survey instrument.

The Utah Education Policy Center (UEPC), under contract with the researchers, administered the survey instrument. Of the 688 invitations to participate in the survey,142 individuals completed the survey. A total of 96 surveys were deemed complete and included in the analysis. Table 1 below describes the demographic representation of the study participants.

Thus, we have two white $\frac{3}{4}$'s and two gray $\frac{3}{4}$'s for a total of four $\frac{3}{4}$'s in three, or $3 \div \frac{3}{4} = 4$.

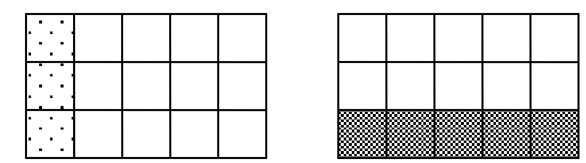
Using area models to explain division of fractions with a whole number dividend is straightforward. However, when the dividend is a fraction, division of fractions with area models becomes more interesting. Let us consider dividing a unit fraction by a unit fraction, say, $\frac{1}{5} \div \frac{1}{3}$: In figure D, the rectangle on the left represents $\frac{1}{5}$ and the rectangle on the right is $\frac{1}{3}$. To count the number of $\frac{1}{3}$'s in $\frac{1}{5}$ we need to

Figure D



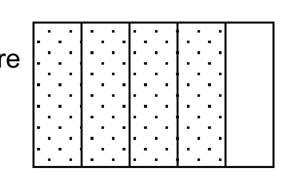
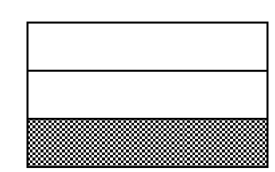
convert each fraction into an equivalent fraction with the same denominator. In Figure E, each rectangle is divided into fifteenths.

Figure E



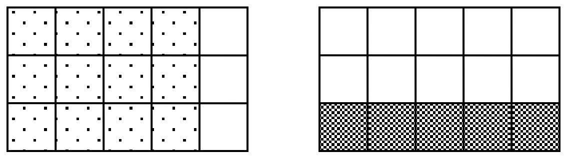
The rectangle on the right tells us that $\frac{1}{3}$ consists of five small squares whereas the left rectangle has three small squares. The question here is how much of a $\frac{1}{3}$ are the three squares that are the $\frac{1}{5}$? Since $\frac{1}{3}$ is made up of five small squares, the three squares from the $\frac{1}{5}$ are $\frac{3}{5}$ of $\frac{1}{3}$. So, there are $\frac{3}{5}$ of a $\frac{1}{3}$ in $\frac{1}{5}$, or $\frac{1}{5} \div \frac{1}{3} = \frac{3}{5}$.

In the next stage, we consider a nonunit fraction divided by a unit fraction, for example, $\frac{4}{5} \div \frac{1}{3}$.

Figure  F  shows the area models for $\frac{4}{5}$ and $\frac{1}{3}$.

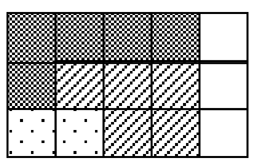
To count the number of $\frac{1}{3}$'s in $\frac{4}{5}$ we divide both area models so that they are made up of equal-sized squares. Figure G illustrates how each fraction is now represented as an equivalent fraction with a common denominator. The model on the right in figure G shows that $\frac{1}{3}$ is equal to five small squares.

Figure G



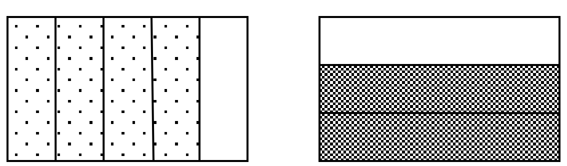
The next step is to find out how many groups of five small squares are in the model on the left. In Figure H, we count off groups of five small squares. Here we have two sets of five small squares with two small squares leftover. So, we know that $\frac{4}{5}$ has at least two $\frac{1}{3}$'s. The two leftover squares represent $\frac{2}{5}$ of a $\frac{1}{3}$, which we know consists of five small squares. So, $\frac{4}{5}$ has $2\frac{2}{5}$ of a $\frac{1}{3}$ in it, or $\frac{4}{5} \div \frac{1}{3} = 2\frac{2}{5}$.

Figure H



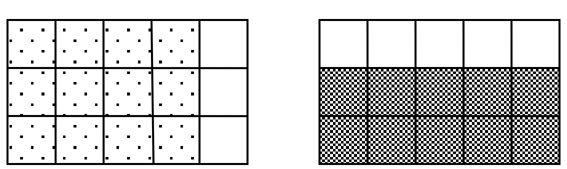
In our final example, we look at what happens when we divide two nonunit fractions, say $\frac{4}{5} \div \frac{2}{3}$. In Figure I, we have area models of $\frac{4}{5}$ and $\frac{2}{3}$:

Figure I



As we did in the previous examples, we divide the two area models into equal-sized small squares to convert each fraction into equivalent fractions (see Figure J).

Figure J



the conceptualization, The paper also presents Utah-specific implications of the research for reconsidering the current structure of mathematics teacher preparation and licensure.

Four Domains of Teacher Knowledge

Teacher preparation and licensure in Utah (as in most states) is framed around two domains: content knowledge and pedagogical knowledge. Augmenting these two domains (content and pedagogical knowledge) to four domains (content, pedagogical, pedagogical content and curricular knowledge) for both secondary teacher preparation and licensure builds on the work of several well respected researchers in the area of teacher preparation and licensure. In the domain of content knowledge, Ball, Thames, and Phelps (2008) provide a structure for understanding how mathematical knowledge for teaching is different than mathematical knowledge for other fields. In the domain of pedagogical content knowledge, research by Hill, Ball and Schilling (2008) frames the complexity of relating mathematical knowledge to the teaching of mathematics. Additionally, Shulman (1986) raises questions about the type of pedagogical and content knowledge teachers need as they enter the profession and also suggests the importance of curricular knowledge for teachers. In terms of pedagogical knowledge, Cochran-Smith, Davis, and Fries (2003) synthesize the current issues in preparing teachers to work with students from diverse backgrounds. Thus, there is a strong research foundation on which to recommend a four-domain structure for preparing aspiring teachers or, as Shulman (1986) says, transitioning individuals from "expert student to novice teacher" (p. 8). This research articulates four distinct components of teacher knowledge. As components of teacher knowledge, all four contribute to quality teaching in the classroom. As a result, teacher preparation and licensure policies that fail to account for all four domains lack the comprehensiveness necessary to adequately prepare aspiring teachers. Each of the domains of teacher knowledge is further described below, followed by further discussion of the proposed four-domain structure of teacher knowledge.

Mathematical Knowledge (content knowledge) is the ability to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, understanding content trajectories (the origin and extension of core concepts and procedures), apply mathematical ideas, use mathematical language and conventions, and reason and do mathematical proofs (Ball et al., 2008; Ferrini-Mundy et al.; Hill et al., 2008). For the purpose of this study, this definition is delimited to the mathematical concepts directly related to secondary mathematics.

Pedagogical Knowledge is the knowledge and skills associated with non-content specific aspects of teaching and learning such as the knowledge and skills needed for teaching English language learners and students requiring special education services, classroom management, multicultural education, learning theory, and motivation strategies (Cochran-Smith, Davis & Fries, 2004; Fennama and Franke, 1992.)

Pedagogical Content Knowledge is the ability to impart mathematical understanding to students and includes the capacity to analyze student mathematical work and interpret what the student does and does not understand; design, modify and select mathematical goals to meet the needs of students in the context of the course; explain mathematical ideas in various manners; deconstruct complex mathematical ideas and attach fundamental meaning to symbols and algorithms in a manner that both maintains the integrity of the mathematics and is accessible to students; and enable students to see and apply content trajectories (Ball, 2000; Ferrini-Mundy et al., 2008; Hill et al., 2008; Shulman, 1986). For the purposes of this study, as in mathematical knowledge, this definition is delimited to mathematics typically taught at the secondary level.

Curriculum Knowledge is the knowledge and skills of the State Core Curriculum (Common Core State Standards or other core/guiding curriculum); programs designed for the teaching of mathematics at different levels; instructional materials available including textbooks, supplementary materials, software programs, and internet tools; and characteristics that serve as both the indication and contraindication for the use of a particular curriculum or program material (Shulman, 1986).

General Interest

The Relationship between Teacher Knowledge, Preparation, and Licensure: A Study of Novice Mathematics Teachers in Utah

__Maggie Cummings, Jordan School District and Karen M. Jackson, University of Utah

One of the primary goals of educational policy is to provide structures that ensure all students have high quality teachers so that students have maximum opportunities to learn.. Teacher preparation and licensure policy have real implications for student academic success (Darling-Hammond & Young, 2002). The research on teacher quality tends to focus on two broad domains, content and pedagogy (Cochran-Smith, 2001; Darling-Hamond, 2000, Darling-Hammond & Young, 2002; Kanstoroom & Finn, 1999; Melnick & Pullin, 2000; Wilson, Floden & Ferini-Mundy, 2001) and teacher preparation and licensure are typically framed around these two aspects of teacher quality. This is certainly the case in Utah. Content and pedagogy, however, are not the only aspects of teaching that matter for teacher quality. Looking specifically at secondary mathematics teachers, this study proposes that two additional aspects of teacher quality, pedagogical content knowledge and curricular knowledge, are important to consider in order to better understand the nature of mathematics teaching and the ways that teacher preparation and licensure policy can be leveraged to ensure that new secondary mathematics teachers are better prepared to meet the challenges of the role.

Recent research from the Learning Mathematics for Teaching (LMT) Project team at the University of Michigan has offered evidence that mathematical knowledge for teaching is more complicated for experienced teachers than the typical two-domain framework of content and pedagogy suggests (e.g. Ball, Thames & Phelps, 2008 and Hill, Ball, and Shilling 2008). Shulman (1986) had previously introduced the concept of pedagogical content knowledge to complement content and pedagogy since it was not completely clear what content and pedagogical knowledge are most important for teaching. Other researchers have argued of late the need to prepare teachers to work with students from diverse cultural, ethnic, and economic backgrounds requires a type of expertise that is not accounted for in descriptions of what teachers need to be able to do (Cochran-Smith, Davis, and Fries, 2003). Taken together, these arguments make clear that content and pedagogy alone do not adequately prepare high quality teachers.

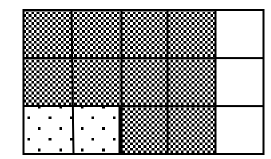
In Utah, as in many states, secondary mathematics teacher preparation coursework emphasizes the two traditional domains of mathematical knowledge and pedagogy. The mathematics courses required for licensure, including content-based pedagogy courses, are outlined by the Utah State Office of Education policy (see http://www.schools.utah.gov/cert/Endorsements/docs/endmath.pdf) and are primarily offered by mathematics departments at Utah's universities and colleges. Coursework in pedagogy is also outlined by the Utah State Office of Education and includes courses on working with students with disabilities, working with students who are English language learners, management, curriculum and assessment, and adolescent psychology; these courses are typically taught in Colleges of Education. Aspiring mathematics teachers must also demonstrate mathematical knowledge by passing either the 0069 or 0061 Praxis exam and demonstrate pedagogical knowledge by passing the Praxis II Principles of Learning and Teaching exam. The coursework and Praxis components of Utah licensure requirements highlight the two-domain structure that underlies the conceptualization of what knowledge is important for teachers.

No research has yet focused specifically on the effectiveness of the Utah mathematics teacher preparation structure in providing quality teachers to Utah's students. This is not to say that teacher preparation or licensure in Utah is sub par. Rather this paper seeks to determine whether teacher preparation in Utah might be adjusted in some manner to better prepare individuals to do the work of teaching mathematics; or if there may be more specific requirements for licensure that might better discern candidates' readiness to enter the profession. To that end, this paper proposes a four-domain structure of teacher knowledge and presents findings from a survey of new math teachers in Utah to test

The right hand model in Figure J tells us that $\frac{2}{3}$ is equivalent to ten small squares. So we now count how many groups of ten small squares are in the area model of $\frac{4}{5}$. Figure K shows us that $\frac{4}{5}$ contain one group of ten small squares with two squares left over. Again, we need to know how much of a $\frac{2}{3}$ do those two small squares represent.

Since $\frac{2}{3}$ is represented by ten small squares, those two leftover squares represent $\frac{2}{10}$ of a $\frac{2}{3}$. So, $\frac{4}{5}$ contains $1\frac{2}{10}$ of a $\frac{2}{3}$ or $\frac{4}{5} \div \frac{2}{3} = 1\frac{2}{10}$.

Figure K



Naturally, our students cannot divide fractions using area models forever and we want our students to understand the standard algorithm of invert the divisor and multiply. So we shift our focus from the area models to the equations that they illustrated, namely:

$$3 \div \frac{1}{4} = 12$$

$$3 \div \frac{3}{4} = 4$$

$$\frac{1}{5} \div \frac{1}{3} = \frac{3}{5}$$

$$\frac{4}{5} \div \frac{1}{3} = 2\frac{2}{5}$$

$$\frac{4}{5} \div \frac{2}{3} = 1\frac{2}{10}$$

In our first example, we saw that $3 \div \frac{1}{4} = 12$. Since $3 \cdot 4 = 12$ and 4 is the reciprocal of $\frac{1}{4}$, we can say that $3 \div \frac{1}{4} = 3 \cdot 4$. Thus we can conjecture that inverting the divisor and multiplying can tell us how many of the divisors are in the dividend. If we invert and multiply the remaining examples, do we get the same results as using the area models?

$$3 \div \frac{3}{4} = 3 \cdot \frac{4}{3} = \frac{12}{3} = 4$$

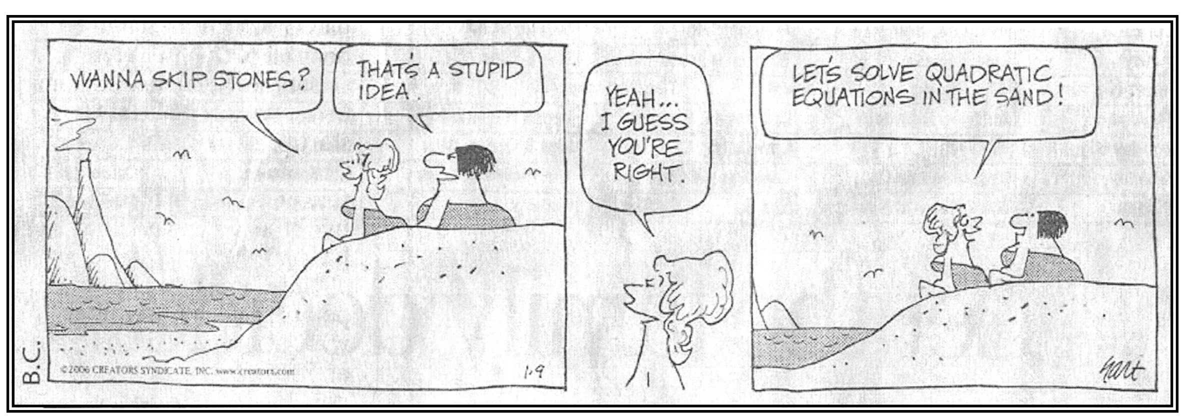
$$\frac{1}{5} \div \frac{1}{3} = \frac{1}{5} \cdot \frac{3}{1} = \frac{3}{5}$$

$$\frac{4}{5} \div \frac{1}{3} = \frac{4}{5} \cdot \frac{3}{1} = \frac{12}{5} = 2\frac{2}{5}$$

$$\frac{4}{5} \div \frac{2}{3} = \frac{4}{5} \cdot \frac{3}{2} = \frac{12}{10} = 1\frac{2}{10}$$

As you can see, application of the invert and multiply rule in the above examples does produce the same results as using area models.

For many students fractions are a stumbling block. Many can learn to manipulate fractions, in particular the invert and multiply rule for the division of fractions. Yet they do not understand why this algorithm works or even what we are looking for when we divide fractions. I believe that area models can help students build a conceptual understanding of the division of fractions.



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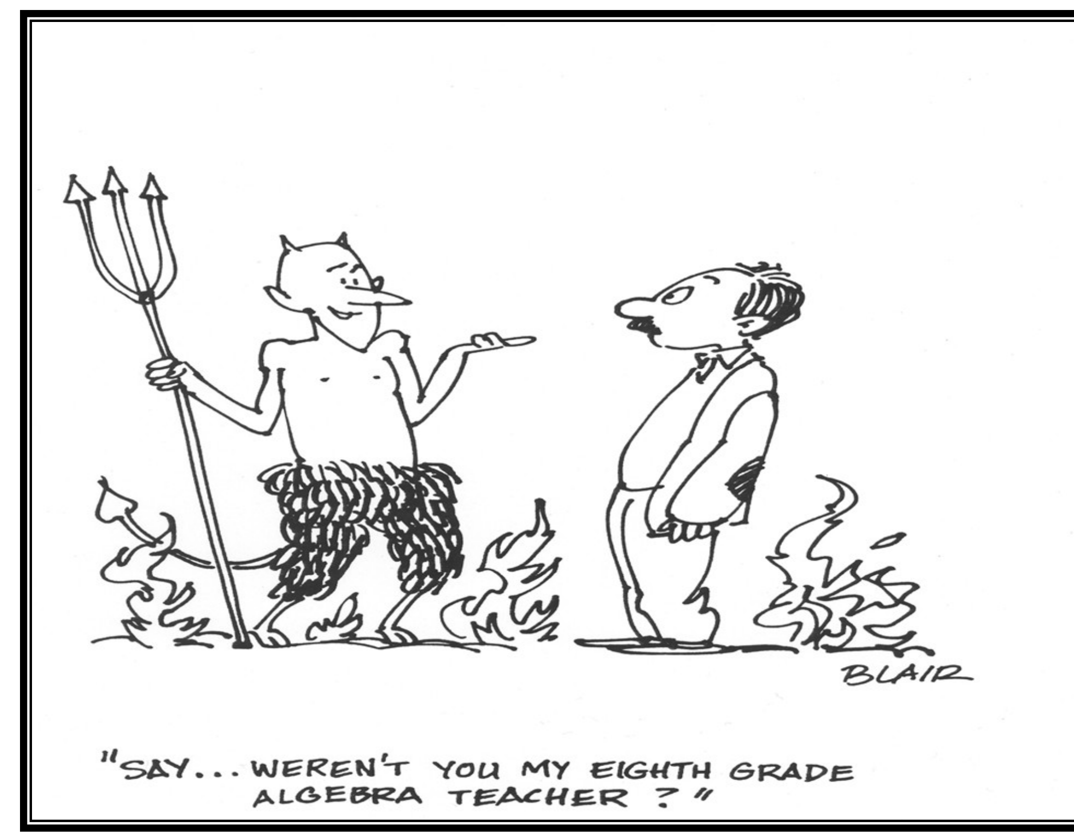

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Grades K-12

Academic Safety

Rachel McAnallen, Ms. Math

Sometimes the most important math lesson is teaching students to respect each other

Have you ever been in a learning situation where you've felt afraid to ask a question?

Recently I was in a classroom where a student worked up the courage to ask a question about the lesson I was teaching. Immediately, I turned to the rest of the class and said, "How many of you are happy that she asked that question?" Half of the kids raised their hands.

"Everyone say 'Thank you, Melissa, for asking the question,'" I instructed. After they thanked Melissa, I went ahead and answered her questions, making sure she was satisfied with my explanations.

Then I stopped the class once more. "Do you know why I told you to thank her?" I asked them. They all sort of looked at me dumbfounded.

"For this reason," I explained. "A good teacher knows when one student works up the courage to ask a question, there are many other students in the class who have the same question. When you raised your hands because you were happy that Melissa asked her question, and thanked her for it, Melissa looked around and saw that she was not alone."

Later that day, I went to the afternoon workshop and one of the teachers commented to me, "I like the way you set up academic safety in your classroom."

Academic safety. I have fallen in love with the term. It is important that we encourage a learning environment in which our students feel safe to ask questions. Melissa took a risk that many kids are not willing to take in the classroom.

One reason students might hesitate to ask a question is a fear that they will get in trouble. Maybe the teacher thought that he or she explained the lesson well enough, and if the learner did not get it, they might be scolded.

To ease this fear, I will often ask kids, "Which would you rather say, "Ms. Math you lost me', or, 'Ms. Math I got lost?"

Of course they always tell me that they would rather say, "Ms. Math you lost me."

And I ask, "Why is that?"

Any they giggle and say, "Well, that way we can put the blame on you."

"That's right," I agree. "You may put the blame on me if you've been listening intently, you've been taking notes, you've been focused the whole time on what I'm doing-then you may raise your hand and say, 'Ms. Math, you lost me.'" Then I will attempt to explain it in a different way, or I may say, "Would a picture help?" I let the students know that I will try to meet their learning style. Because saying it louder and slower the same way isn't helping a student if they're lost.

But I believe the most common reason students don't feel safe asking a question is because they are afraid their classmates or the teacher will think they're stupid, and they are afraid of being ridiculed.

I was in another classroom where a student asked a question and some other people laughed at her. Once again, I stopped the class immediately. As I had done in Melissa's classroom, I asked how many of them were glad this student had asked the question. Almost everyone in the class raised their hands.

Then I asked, "How many of your make fun of people?" Hardly anyone raised their hand. One hand that was noticeably absent belonged to the boy that laughed the loudest at his fellow classmate only a minute earlier. "Stop that!" I said. "You're sitting there not telling a truth…you guffawed the loudest when Natasha raised her hand. In my class you will not make fun of anyone who asks a question, or who answers incorrectly."

It is my belief that we need to establish academic safety at all grade levels, beginning in Kindergarten. The instant that ridicule becomes an issue in my class, I let students know that it isn't acceptable to make fun of someone who wants to learn. Then I tell them, "Now if this occurs in class again, I will point to the person that makes fun or laughs and say…'that's not academically safe.'"

It changes the whole demeanor of the classroom. More kids-especially girls-will start to raise their hand. In my heart of hearts, I believe if we made every classroom academically safe, we wouldn't have to worry about physical safety out on the playgrounds.

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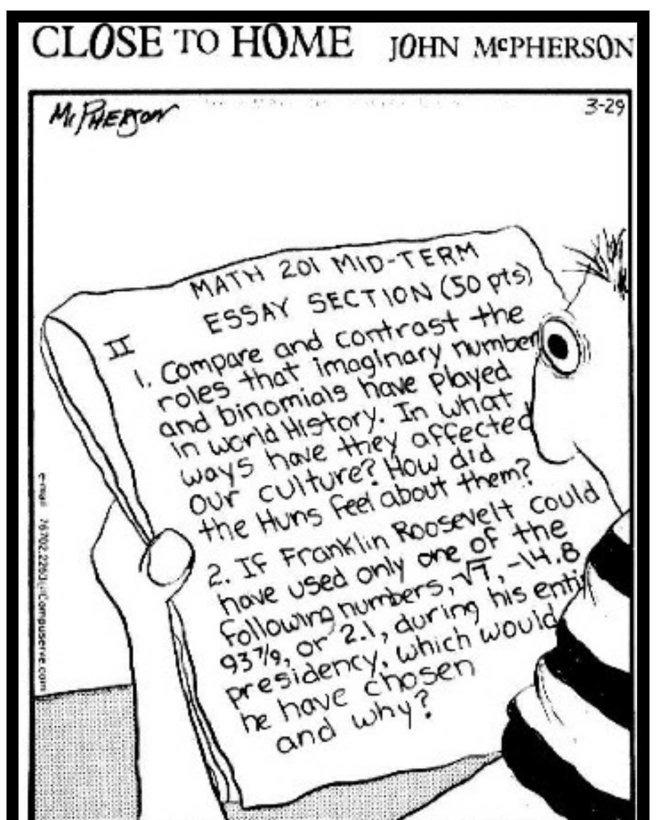

A diabolical new testing technique: math essay questions.

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3) coherent challenging lessons that make the
4) mathematics meaningful to students by
5) well trained and knowledgeable teachers that use
6) classroom time effectively (by limiting unguided seatwork and review) that
7) motivate students and help them understand that personal success in school (and in particular mathematics) is based more on effort than on natural ability.

To accomplish these outcomes wise use of resources is necessary with schools and teachers supported by policies and a support system that helps them to maximize student learning. Such policies usually include a

8) focused, clear and explicit (national) curriculum that illustrates in detail what students should know at each grade and how to know if they have adequately mastered the curriculum;
9) reasonable accountability measures for schools and teachers that focus on more than just student achievement scores;
10) limiting non-instructional staff and administrators to keep the best teachers in the classroom teaching as many kids as is feasible and to keep as many resources as possible tied to improving instruction; and
11) recruiting teachers from the top of the academic distribution.

Each of these statements are easy to make in general, but the hard task is finding ways to move from where we are now to where we need to be in efficient and effective ways. This will be hard work that may require conversations across many interested parties. I hope that this report will help in facilitating these conversations.

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Grades K-6

Personal Lenses: The Nature and Utility of Oral Retellings of Word Problems by Third Grade Students

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Rationale: Discerning A Need for Lenses

A Need

Traditionally mathematics has been taught as a collection of skills, rules, facts, symbols, and formulas to be memorized (National Council of Teachers of Mathematics [NCTM], 2000). But in today's world, with its growing dependence on technology, citizens must reach beyond memorized rules, processes, and symbols to access scientific and mathematical thought. Students must learn to solve real-world problems by discerning patterns, making hypotheses, forming conclusions, and communicating (e.g., Baroody, 1993). As they practice and master these mathematical processes, their capacities for solving problems and thinking logically increase, and they learn to communicate with precision (Utah Mathematics Steering Committee, 2009). Personal as well as economic and career success (National Mathematics Advisory Panel [NMAP], 2008) are thus affected by learning to think mathematically rather than merely respond.

A strong body of research demonstrates that skills and strategies are more meaningful for children who perceive them as relevant to aspects of their daily lives and can relate them to prior experiences (e.g., Barnes 2008; Blachowicz, 1994; Hardman, 2008; Silliman & Wilkinson, 1994), and that oral language is an important medium in allowing them to do so (Naremore et al., 1995). Since well-constructed word problems connect math with situations children commonly experience and use language as a medium for comprehending and thinking, they are a natural format for mathematics instruction.

A Challenge

But for many children, word problems do not seem natural at all. Much of the difficulty children have with word problems may be related to the language with which those problems are expressed. The language of word problems is different from the language children use in describing their own problems and experiences (Pimm, 1987; Kliman & Richards, 1992). Not only is the vocabulary different (Paramar, Cawley, & Frazita, 1996; Allen, 1990), the semantic structure of the problems is difficult for many children to access (LeBlanc & Weber-Russell, 1996).

If unfamiliar language makes a problem difficult to understand, children cannot fully apply the complex thinking that may be necessary to solve it. Just as unfamiliar language can blur meaning, the natural, comfortable language of one's daily life can clarify it. Douglas Barnes (2008) explains that using children's spontaneous language allows them to retrieve and transform what they know—"a crucial part of learning" (p. 3). They "talk themselves [into] understanding" (Barnes, 1976, p. 108 as qtd. by Solomon & Black, 2008, p. 74).

A Metaphor

Whitin and Whitin (2000) express the relationship of language and complex mathematical thinking: "Writing and talking are ways that learners can make their mathematical thinking visible" (p. 2). This linkage of language and visibility leads to a metaphor that unifies many of the ideas asserted in the literature and clarifies findings in our study, some of which we anticipated and many of which we did not.

Retelling a problem in one's everyday language is like putting on a pair of individualized eyeglasses. Individual vocabulary and language structure, like individual lenses, are derived from, adapted to, and utilized to meet individual needs. No one would ask a child to read wearing the teacher's glasses, yet many of us expect the child to understand a word problem using the teacher's language. Even if the lenses are designed to provide similar information, they are not natural or comfortable—they don't focus accurately because they don't fit. When children retell a problem in their own words, they are able to see the problem through lenses that fit them and are appropriate for their needs.

Our Study: Making Lenses Available

Antecedents

As they orally retell, children adjust the language of the problems to be consistent with their own formats (Vershaffel, 1994). This process has been shown to affect children's mental structures (Pickert & Chase, 1978) and reconstruction of meaning (Barnes, 2008; Scott, 2008; Teale, 1984). In discussing the effects of different kinds of talk in science classrooms, Scott (2008) calls attention to the distinction made by Vygotsky between *everyday* and *scientific* talk. Everyday talk is the language of "informal interactions of day-to-day living" (Scott, p. 18). Scientific language "relate(s) to disciplinary knowledge" (p. 19). Primary grade children need the everyday lenses to help them cross the gap to the scientific, disciplinary side.

Few studies have applied the oral retelling strategy to mathematics word problems; however, Cai, Jakabcsin, and Lane (1991) did find that implementing oral retellings produced "significant learning" in the aspects of "comprehension and recall" (p. 257). The full potential of this type of adaptive lens has not been established for mathematics, but we do know that children are seeing better, and we have basic ideas as to why.

Expectations

Our treatment expectations were built on our knowledge of the power of language to enhance comprehension, internalization, personalization, and application—processes basic to thinking mathematically in real-world contexts. Countryman (1992), a noted mathematics teacher, acknowledged the fundamental connection: Words are instruments that facilitate thought.

Our Examination

To explore the effects of oral retelling on third graders' ability to understand and solve word problems, we addressed the following questions:

1. After instruction and practice with oral retelling, to what extent do third graders use this strategy in solving word problems when they are not required to do so?
2. What is the nature of the oral retellings that third graders use in solving word problems?

In terms of the metaphor, (a) are children actually wearing the glasses that have been provided, and (b) how are they wearing them?

Procedures

The study consisted of pre-treatment interviews, a sequence of lessons, and post-treatment interviews of the same children. Eight students were chosen as the focus for the study, but the entire class was observed and video taped during the lessons. The focal students were selected by their teacher as representative of the range of mathematics abilities and backgrounds of the students in the class.

Pre-treatment Interviews

Before the study began, interviews were conducted during which the eight focal students were asked to solve several multiplication and division word problems, with low-range numbers, and then urged to explain their thinking. No prompting or scaffolding was used, consistent with Morrow's (1988) recommendation that such aids may be included in instruction but avoided in assessment (as cited by Bernfield, Morrison, & Wilcox, 2010). We checked the children's vision without the lenses we planned to introduce.

There is no Silver Bullet. There are multiple factors that lead to student outcomes both within and without of school. No single change can address all of the mitigating factors that lead to the problems we are facing in mathematics education and in education in general. Educational research has largely been a story of failed attempts to improve education that have ignored the complex nature of schooling and the difficulty of changing the cultural factors of schooling (Cohen & Ball, 1999; Stigler & Hiebert, 1999). Where there has been large scale success it is because improvement efforts are *sustainable regimes of improvement efforts coordinated to improve and support high-quality academic work.*

A regime is a set of particular interventions or policies that work together and support each other to overcome the problems of a particular school or school system. Simply testing teachers, requiring high-stakes tests, raising salaries, changing curriculum, mandating more teacher professional development, or reorganizing school administration will probably have little overall effect by themselves, but each of these could be an important part of an effective regime.

If you want to effect what students learn in school, you need to change what they experience in classrooms. This may be an obvious statement but this is also another key reason for the history of failure of educational reform. Many of the educational policies and programs have not focused on the classroom and as such have made very little or no effect on achievement. Students and teachers often just keep on doing what they are doing as many policies, standards, and programs change around them. Policies dealing with school funding, teacher certification, or high-stakes testing, for example, rarely deal with reasonable assumptions about how such policies will impact learning because their effects are mediated by school administrators, teachers, and students. Efforts that start with changing what students experience in the classroom have a much better chance of making an impact.

*Students can learn much more mathematics then we currently require. They can also learn the mathematics better.*All of the evidence from international studies, national studies, and exploratory studies in mathematics education show that US students, and in particular, Utah students, learn much less mathematics than they are capable of. To fix this will take a combined effort on the part of parents, students, teachers, administrators, and legislators. Many US citizens are satisfied with mediocre results on a relatively unchallenging curriculum.

Students must practice the skills in school that we require them to have outside of school. The skills needed to succeed in college and the job market, particularly the job market of the future in the US, require the ability to perform non-routine tasks that require extensive problem solving skills and the ability to communicate their ideas in a clear and concise way that justifies their thinking. Typical students are rarely asked to perform such tasks in K-12 education. Helping students to engage in these kinds of tasks will greatly help students build the skills they need for future work, in and out of school. Currently there is a large disconnect in the low-level routine tasks students do in school and the high-level non-routine tasks that require the ability to read critically, write persuasively, and reason analytically.

Conclusion

It is clear that we are losing our leadership role to other industrial countries when it comes to learning mathematics. This is especially true by the time our students reach 12th grade where they drop from about average to near last in the international pack in the four years of high school. Our highest achievers have a hard time competing with the top performers of the top countries. This lack of achievement costs the US hundreds of billions of dollars a year in economic growth in the form of money spent on remedial education (both by businesses as well as schools and universities) as well as lost growth due to a lack of human capital. This does not include the jobs lost to other countries where there are an abundance of highly qualified professionals.

Improvement is possible at many levels. Is there one quick fix, no. But these examples, and others, show that the following are general characteristics of effective schools:

1) concentrated effort focused on
2) improving the quantity and quality of mathematics instruction with

commitments. Many of these characteristics are those shared by schools in high achieving countries.

How well do these charter schools raise student achievement? One of the most rigorous studies has been done in New York (Hoxby, 2007). The state law in New York requires charter schools to select students at random if there is not enough room for all applicants to fit the open positions. This allows researchers to compare the students who randomly got in to those who were randomly left out, thus eliminating selection bias.

The results found by Hoxby (2007) show that charter school students have much larger gains, on average, than non-charter school students whose parents had applied to charter schools. Not only do charter school students gain compared to their peers, low-income urban students, but the first six years of data show that the average charter schools put students starting in first grade on pace to catch up to the average student in the richest New York suburbs by the time they graduate from high school. The study is continuing to see if the charter schools can actually keep up this dramatic pace.

There are many stories of teachers dramatically improving a classroom and helping students learn, but this following example shows the possibilities, even within the current system. Michelle was a Teach for America teacher in an inner city public school in Baltimore and was assigned a second grade class. This particular elementary school tracked students by behavior as well as achievement. Her class was supposed to hold the lowest achieving students with the worst behavior problems. In her own evaluation, she failed miserably. But she did not want to be run out of town by a class of eight year olds. She spent the following summer preparing a rigorous curriculum and plan that could help these children succeed. She knew that these kids were well behind where they should be in second grade and would have to do more than the average kid to catch up. She helped the parents of her students to understand what she was undertaking and what to expect in terms of the amount of homework, and if needed extra time at school, to succeed at their studies.

Michelle went one step further and decided to team teach with another teacher to combine their strengths. Since they each had responsibility for a class this created a class of 70 students. Michelle and her team teacher taught the same set of students for two years in a row. A process called looping with students.

The results were very different from Michelle's first year. The 70 students started on average at the 13 percentile in the nation. By the end of the two years 90% of the students were scoring at the 90th percentile or higher (Rhee, 2009).

The reason I point to this story is that from Michelle's first year to her second (and third) year many things did not change. Her students' parents, socio-economic status, race, diet, neighborhood, health, local crime rate, etc. all remained basically the same. The school principal, district policies, state testing regime, funding, etc. also did not change. What changed was the quality of instruction, the expectations of the teachers, and quantity and quality of the work in which the students engaged. Michelle herself pointed out that it was the adults in front of the students that changed. Not changing from one adult to another (because it was Michelle both years) but what the adults did changed dramatically.

Although these examples were not all math specific, a book that illustrates some of the issues and success stories in mathematics is called "What's Math Got To Do With It?" by Jo Boaler. It provides a further overview of many of the issues in mathematics education, some of them have been touched on in this report.

Principles of Educational Improvement

Because everybody has had years of experience in school and some have had years of experiencing their children go through the schooling process, it is hard to find anybody that doesn't have some insight that they can offer about the problems of schools and a possible solution. This section of our report focuses on key principles to understand about educational improvement that are important to keep in mind when discussing potential avenues for improvement.

Lesson Sequence

A sequence of fifteen 30-40-minute lessons was conducted over a 4-month period, giving students time for review and practice of oral retelling. As Morrow notes, retelling is not an easy or a natural process (2005), but students find it easier and become more proficient with instruction and guided practice (1985). Most lessons began with a brief review of why and how to use oral retellings, followed by a "problem of the day," which students practiced retelling as a whole class and/or in small groups or partnerships. Vygotsky (1978) proposed a view of language as a cultural tool for people to "think together" and "try out ideas" (as qtd. by Mercer and Dawes, 2008); we were anxious for students to have experience with this tool.

As Mercer and Dawes (2008) extend, during conversations or small group sessions speakers frame their "tentative thoughts" (or tentative retellings), and feedback from others contributes to their processes of developing and clarifying (p. 66). Classroom observation during our research showed that more children were willing to retell for the smaller audiences than for the full class; also the classroom teacher, the researchers, and the research assistants were able to offer scaffolding and encouragement, as the lessons were learning experiences--not assessment As children compare and contrast what they "see," all are able to see it from different angles and perspectives.

After the students had retold the problem several times, they were challenged to solve it in at least two different ways--sometimes individually and sometimes in partnerships that gave them further opportunities to use oral processing in their collaboration. At the end of the lesson two or three students would be allowed to present their solutions to the class, illustrated on an overhead transparency and followed by a class discussion of their strategies. Thus we offered them several types and intensities of lenses and helped them adapt to the potential for seeing the problem context more clearly and widening their view of possible meanings as well as alternative solution strategies.

Post-treatment Interviews

Interviews were conducted with the eight focal children 5.5 months after the original interview sessions. Students were asked to do the following: (a) explain what strategies they generally used in solving word problems; (b) solve several multiplication and division word problems, with low- to medium-range numbers, and explain their thinking; and (3) orally retell at least one of the problems before solving it. We tested the vision again, with the glasses sitting on the table. Interviews were audio and video taped and transcribed for analysis.

Results and Conclusions: Discovering a Different Kind of Lens

Question 1: Ability and Choice.

When they were requested to retell a specific problem orally, 6 of the 8 students interviewed were able to do so completely and accurately and subsequently to solve the problem correctly. Thus they were able to use oral retelling as an isolated strategy. This process may have helped them in solving the word problem; they claimed that it did. Most of those interviewed stated that they used oral retelling in solving word problems, particularly for difficult problems. However, in solving problems during the interviews, they did not use oral retelling unless they were specifically asked to do so.

Question 2: Nature of Third Graders' Oral Retelling.

Although they did not spontaneously use complete oral retellings, third graders did frequently use what we have referred to as *partial retellings*. Often they would repeat the numbers given in one or both conditions, sometimes giving the nouns as well: "10 rows of flowers," "five cookies in six bags." Some restated the problem question; others framed their own questions to clarify one or both conditions or to check their understanding of the problem question.

The children seemed to be using language to confirm or explore aspects of the problems, but their statements were less complete and less sophisticated than those designated as "oral retellings" in the literature related to either mathematics or language arts.

If an assessment instrument such as the Reader Retelling Profile (Morrow, 1988) or its parent document the Richness of Retellings scale (Irwin & Mitchell, 1983) had been used, none of the retellings would have been found to meet the criteria for retelling use. This spontaneous use of language did not seem to be related to the formal instruction the students had received in oral retelling, as neither the frequency nor the sophistication of these partial retellings increased from pre-treatment to post-treatment interviews.

This use of partial retelling appeared to be subconscious as well as spontaneous. When questioned specifically about their use of oral retelling, the children who claimed to use the strategy stated that they used it with difficult problems. Yet the use of spontaneous partial problem retelling did not appear to be related to the difficulty of the problem or, somewhat ironically, to the success of the solution. As researchers we were puzzled that the instruction, guidance, and practice in retellings seemed to have had no effects on actual student performance in solving word problems. In an earlier study (Cutler & Monroe, 2006), sixth graders had found oral retelling to be a strategy that is easy to learn and helpful to use in solving word problems; and they used the strategy spontaneously once they have mastered it. Third graders, however, did not. Students seemed to be carrying their glasses but not using them.

What Might We Learn from These Findings?

As we studied transcripts of third graders' partial retellings, we noticed that the use of language was spontaneous; children seemed to do it almost without being aware of what they were doing. They certainly would not have identified their utterances as *oral retellings*. Yet the words and phrases they uttered were aspects of the problem they needed to solve, and they were using language to identify them and perhaps sort them in some ways in their minds. To our knowledge, these children had not been previously taught to use oral retellings, particularly in mathematics, and fairly complete summarizing was possible but not automatic for them.

In revisiting the study later after studying the pioneering work of Douglas Barnes in the area of exploratory talk, we noticed a connection that we thought might be applicable to these third graders' partial retellings. Barnes (2008) states,

> When young people are trying out ideas and modifying them as they speak, it is to be expected that their delivery will be hesitant, broken, and full of dead-ends and changes of direction. This makes their learning talk very different from a well-shaped presentation. (p. 5)

Barnes contrasts this exploratory talk with presentational talk, which presents ideas that are more carefully thought through in more sophisticated organization and language. A full retelling is at least somewhat presentational; a partial retelling is exploratory. Mathematics education researchers Solomon and Black (2008) apply this concept: "Pupils frequently guess what teachers want them to say in pursuit of providing them with 'final draft' answers" (p. 77). Older students are likely to have the sophistication to recognize that if they are being interviewed by a researcher who has been teaching them and is currently asking them about oral retellings, they are expected to produce oral retellings; and they are able to produce an acceptable final draft. These third graders were still in rough draft stages, but they were using oral language to explore.

Conclusion

This application of the concept of exploratory language to our findings on partial retellings is our own. We recognize that only 8 children were pre- and post-tested and that they were teacher-selected "representative" students from one classroom. This study was exploratory; further research needs to be done with larger and more randomly chosen populations. Additional research might also make direct comparisons between exploratory language used in mathematics and the more commonly recognized and examined exploratory language on literacy or social studies topics.

The children we observed and interviewed chose lenses adapted to the vision that they needed and frames that fit their own faces. Adult glasses seemed to be of little value to them. And glasses made for 12-year-olds were neither comfortable nor easily used by children who were only 8.

increase in China is still greater both in number as well as percentage. In 2003 the US graduated twice as many students from college as China, in 2015 it is predicted that the roles will be reversed with China producing twice as many college graduates as the US. With the rise in advanced skills in China (as well as in India although we don't have good estimates for India college graduates), it is little wonder that China has been doing well economically. Remember that money flows towards advanced skills and China will soon have more total college graduates than the US. This will cause a greater flow of money towards China then even the current rate, and much of that money will be drawn from the US economy. Not incidentally a major reason that the US became the major world economy in the first place is that we were the first major country to invest in universal education for all and successfully help a large majority of the population (in some states over 90 percent) graduate with a high school level education.

Table 1. Estimates of High School and College Graduates in Four World Regions (from Schleicher, 2007)

Year	US	China	European Union	India
High School Graduates				
2003	3,000,000	7,000,000	4,000,000	4,000,000
2015	4,000,000	13,000,000	4,000,000	11,000,000
College Graduates				
2003	1,850,000	950,000	1,800,000	NA
2015	2,500,000	5,000,000	2,500,000	NA

The trend does not just hold in China, but countries around the world have a greater percentage of students getting college degrees than the US. In 1995 the US lead the 22 strongest OECD countries with a college graduation rate of about 34%. In 2005 we ranked 14^{th} and our overall graduation rate stayed about the same (Schleicher, 2007). Australia, for instance, was ranked fifth in 1995 with a college graduation rate of about 24%. They now lead this group of OECD countries with a graduation rate of nearly 60%!

Where is the Hope?

Many people in the US may be frustrated at the lack of measurable improvement in student learning. But there is some good news. There is ample evidence that improvement can be made at the classroom level, the district level, and the state level.

Minnesota is a state that has made dramatic improvements over the last decade. It competed in the TIMSS study as a separate benchmarking state and scored very well in the 2007 TIMSS. Minnesota is not the highest achieving state in the country (currently that belongs to Massachusetts) but Minnesota's gains in achievement scores is very impressive. Minnesota went up 38 (US went up 11) points to a score of 554 in the fourth grade TIMSS score from 1995 to 2007. Minnesota made some dramatic changes in that time. The state adopted a coherent, focused, grade-by-grade math curriculum developed by a team of Mathematics educators and curriculum specialists. The curriculum focused on overcoming many of the curriculum and instructional concerns raised in this report.

Any state could follow Minnesota's lead by implementing a well-designed, coherent curriculum in mathematics. It allows teachers to know what is expected and to be supported by appropriate training and resources to enable their students to reach world standards.

KIPP schools, and similarly structured charter schools, have also shown to dramatically improve student achievement, especially in lower-class urban schools. KIPP, which stands for Knowledge is Power Program, runs charter schools according to their model which is based off of very high expectations, tight discipline, rigorous curriculum, ample learning time and explicit parent and student on five types of skills. The trends clearly show that the national reports have correctly forewarned about the job market changes. Routine manual jobs have been decreased due to better machines and manufacturing technologies. Non-routine manual jobs, have dropped and leveled off as increases in technology have reached the edge of their impact on these jobs. Tasks such as driving a bus or wiring a house are difficult to automate. Jobs based on routine cognitive skills have been superseded by use of computers. It is the non-routine analytic and non-routine interactive skills based jobs that are seeing growth. The shocking fact about the trends in US jobs is that demand is drastically dropping for the skills that are most easily taught and learned by students. This creates a dilemma for schools because the skills that are easiest to teach and test are also the ones that are easiest to digitise, automate, and outsource.

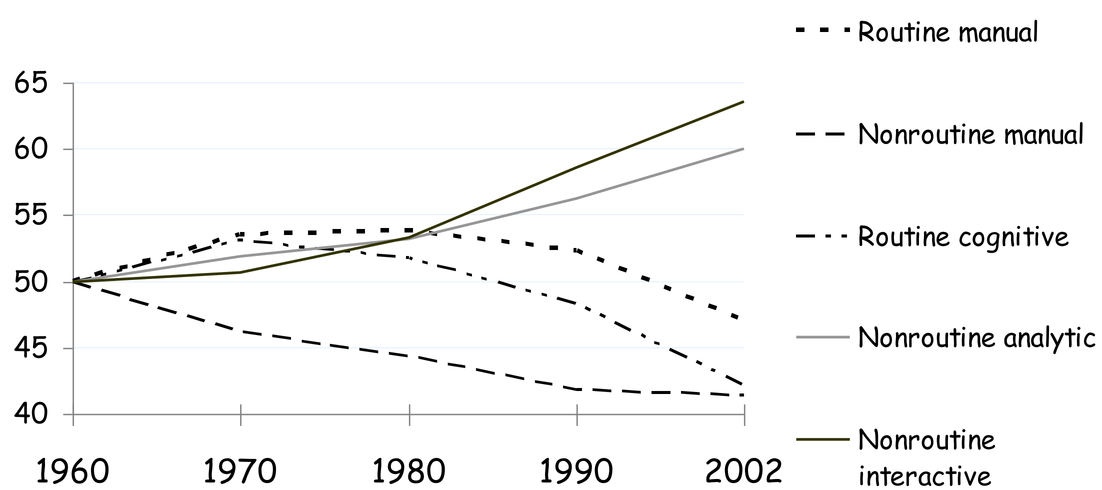

Figure 2. Trends in US jobs based on primary required job skill.

Trends in International Competition

The US economy depends on a qualified and educated workforce, with a high number of high school graduates and a high number of college graduates with advanced skills. It is natural to think that there would be *qualification inflation*, the idea that if everybody had a college degree they would be of little use, and they would be devalued to that of a high school diploma. The result would lead to college graduates staffing all low-level routine and manual jobs and the economy would be little affected. This does not tend to happen.

The evidence gained so far all points to a dramatic increase in growth if all citizens of the US were to get college level training and advanced skills. In short, money flows towards advanced skills. This works internationally as well as individually. If the US students had in recent years reached the level of the current world leaders, the 2008 GDP would have increased $1.3 trillion to $2.3 trillion dollars, or an increase of 9% - 16% (Mckinsey & Company, 2009). Eliminating the gaps between African-American and Latino students and white students would have also made a dramatic difference in the US GDP, $310 billion to $525 billion dollars (McKinsey & Company, 2009).

It is important to understand that money tends to flows towards valuable skills because the world wide distribution of valuable skills is dramatically changing. Table 1 illustrates the dramatic change that is taking place in the world with respect to academic qualifications. From 2003 to 2015 the US has (and will have) an increase in the number of high school graduates, but that increase is shadowed by the increase in China and India. The data on college graduates shows an even greater percentage increase in college graduates for the US as compared to the high school increase. Unfortunately the

Use of language in clarifying and solving word problems must be appropriate for the specific group of students. One size definitely does not fit all.

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General Interest

The State of Mathematics in the United States: An Overview

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In a previous article I describe the state of mathematics education in Utah, in particular with how Utah is doing now compared to how we have done in the past as well as how we are doing now compared to other states (Corey, 2009). In this article I describe the state of mathematics education in the US, and at times make some particular connections to Utah. An important source of data in understanding the current state of mathematics education comes from comparisons to other countries, generally higher achieving countries. I certainly can not cover every aspect of the international studies done in the last few decades, but I do try to take some of the most important findings so as to better understand how our education system differs from other high achieving countries, as well as to point to areas that stakeholders may be able to focus on to make improvements. The article ends with some important points to keep in mind when considering changes.

My experience has shown that when sharing results with international studies with parents, teachers, administrators, or other stakeholders (especially results that show that the US is doing poorly for the resources which we have) that there are two types of reactions. The first type of reaction is the *rationalization response*, finding ways to justify our poor showing or to explain why the US can not compete with these other high achieving countries. Common reasons include differences in culture, diversity of students, parent support, teacher pay, un-motivated students, English Language Learners (or second language learners), over-importance put on exam scores, psychological damage caused by pushing kids too hard, etc. The list could of course go on and on. This report illustrates that many popular reasons for our poor showing are not true and although there are differences in these areas between the US and some high achieving countries, many higher achieving countries have these educational or social problems to greater extent than the US. We are unjustified, then, in using them as a reason for poor performance if they do not prevent other countries from performing well. Of course social and cultural differences can and do produce differences in educational performance (some of which are explored in this report), but careful consideration should be taken before attributing differences to purely social or cultural differences.

The second type of reaction is the *copy response*, suggesting that if we adopt a practice or materials of high-achieving countries that such adoptions will improve US education. This follows from the lets-just-do-what-they-do logic. Lets use their curriculum, lets train our teachers using lesson study (a form of training practiced in Japan), lets structure schools like they do in Finland, lets have students use calculators like they do in the Netherlands, or lets lengthen the school year or school day like Taiwan. Would the Netherlands (one of the highest achieving European countries) match the top country in the world if they just imported their curriculum? Would Japanese students be able to learn as much science as students in Finland if the Japanese school system was structured like the one in Finland? Of course the answer to these questions is no. It would take much more than these single changes, yet that is often the suggestion for fixing the US educational woes. It is not just a countries curriculum, or teacher training, or school structure, or student incentives, or length of time in school that determines what students experience in school (and subsequently what they learn). These and other factors interact in complex ways to support or inhibit student learning. High achieving countries have been able to produce and sustain high-levels of learning because they have a whole system of education with policies, training, materials, culture, and methods that work together. We can certainly learn from other countries and borrow ideas and materials, but we also need to be very sensitive that the effectiveness of any practice or material implemented here in the US will be mediated by other factors.

Even though I have not yet shared many results from the international studies the reader may sense that the news will largely be disappointing. Please don't dismiss these results and say that our national education system (or our states system or our school district or our neighborhood school) is doing just fine and that there is no need to worry.

above the national average, and would make the average Utah starting salary rank about 21st in the country (Morgan & Morgan, 2007).

Effort vs. Ability

When it comes to learning mathematics or succeeding in school in general there is a major difference in the belief about what it takes to succeed between the US and some high achieving countries. Students, parents, and teachers in high achieving countries tend to believe that all students can achieve the learning objectives and succeed in school if they put forth the effort. They acknowledge that some students may need to put more effort into their work to succeed than other students. In the US many students, parents, and/or teachers tend to believe that to succeed in school, and in particular in succeeding at learning mathematics, students need to have a natural ability for schoolwork or "thinking mathematically" (Stevenson & Stigler, 1992). There is plenty of evidence from research that all students can succeed at learning mathematics when they are taught well, are intellectually engaged in solving problems, and work hard at learning mathematical skills and strategies (Hiebert & Grouws, 2007).

Erling Boe at the University of Pennsylvania searched for reasons for the differences in achievement between countries in the TIMSS studies and looked at the relationship between a countries average score and the average amount of a subsequent survey (of 120 items) completed by students (Boe et al., 2004). Students were given plenty of time to finish the survey, so students that didn't even answer questions had mainly stopped trying to answer questions. Thus, the percent of the survey completed represented how hard students would work on doing an exam that would have no impact on their grade. The correlation between the country score and country completion rate is very high, above .9. This gives evidence that high achieving countries had 1) students willing to put forth their best effort, and 2) students with integrity to work hard, even when it didn't seem to immediately benefit them. Working hard and always doing your best can be taught by parents and teachers, and is already being taught by many.

Clear Combinations

The latest PISA study focused on attributes of schools and schooling systems that separated the highest achieving countries from the middle and lower achieving countries (OECD, 2007). The first thing that they noticed is that some attributes alone yielded little return, but when combined with another practice or attribute showed great returns (Schleicher, 2007). Here are two pairs of characteristics, that when both present, were associated with high levels of student learning. First: Very high ambition and expectations for their students COMBINED WITH Strong support systems for teachers and schools to meet the standards and expectations. Both of these characteristics needed to be present. Denmark has very good support systems for teachers, but compared to other countries were weak in the level of content they expected their students to master. Other countries were the opposite.

Second: Autonomous schools COMBINED WITH intelligent school accountability and interventions for struggling schools. High levels of school autonomy in the management of the school (hiring and firing practices, teaching methods, etc.), to achieve the national standards and to help students learn the set forth curriculum allow schools to make local, timely changes to overcome problems and improve the learning environment and experiences of students. In Finland, the highest achieving country in this study, every public school functions with freedoms similarly to US charter schools, and in some areas more freedom than US charter schools. In France, even private schools have extremely high controls on them so they are more "public" than public schools in Finland. This school autonomy benefits students the most when there are accountability measures in place and effective interventions for schools that fail to meet the accountability standards. Not all accountability standards are based on bottom line test scores though (and in some high achieving countries testing plays a much less important role than it currently does in the US). Other measures accountability measures include instructional quality, content coverage, and other student outcomes.

Job Trends

National reports in the last decade have warned that the job market of the future is changing and requires skills different than those focused on in K-12 and, in many cases, collegiate education (Glenn, 2000; National Research Council, 2001, OECD, 2006). Figure 2 shows trends in jobs based

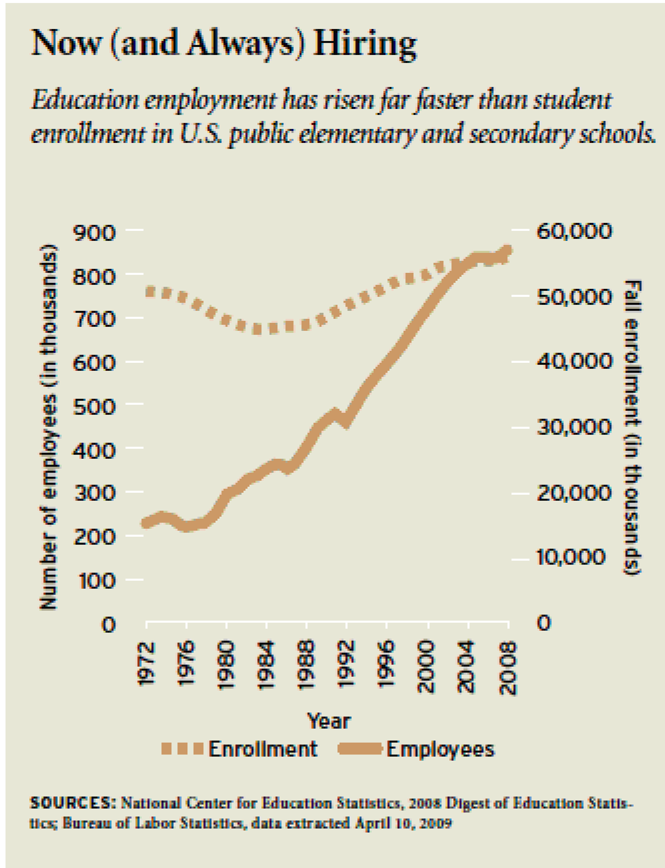

Figure 1. Employee and Enrollment Growth for Public US schools since 1972.

As discussed earlier in the report, studies suggest that high achieving countries try to minimize the number of full time administrators in schools. They focus on maximizing employees that make the most difference for students and ensuring they are well trained and satisfied with their jobs so they can keep the best teachers instruction students. This is the opposite direction that the US seems to be heading. Our school systems are becoming more top heavy (with administrators) and bottom heavy (with support staff). Administrator positions are often filled by the better teachers so with more administrative positions the greater the pull for good teachers to try to move up for better paying positions (Sowell, 1993).

The burgeoning number of employees is one reason that we have average paid teachers in spite of per-pupil funding that is top in the world. The money is spread out to so many employees, including highly paid administrators, that less money is left for teacher salary, benefits, and training. A more efficient system that relied on fewer administrators and support staff could free up large amount of monetary resources to focus on improving classroom instruction for students. States vary widely on the amount spent on administrators and support staff. States also vary on the number of non-instructional staff. Teachers in Utah make up about 49.9% of public school employees, this ranks 30th in the nation. In the top five states teachers make up at least 63.6% of public school employees (Snyder et al., 2009). To reach the level of the top states Utah would have to eliminate more than forty percent of non-instructional employees. An equivalent statement, although it sounds much worse, is that a top five state would have to increase non-instructional staff by more than 70 percent to reach the current ratios in Utah. If Utah reduced its spending on administrators and staff by sixteen percent (about five percent of the total state educational budget) then enough money would be freed up to give teachers an across-the-board raise of more then six thousand dollars. This would place teacher average salaries in Utah

It may not be surprising to know that the level of satisfaction of a parents of a country is highly correlated with the achievement of their students (Organization for Economic Co-operation and Development (OECD), 2007). But readers may be surprised to know that the direction of the correlation is negative! The more parents are satisfied with their school system the worse the education system tends to be. Although various reports and articles have talked about the generally poor condition of US schools, many people are satisfied with our nation's schools or our state's schools, and that is one reason why our schools are doing so poorly and why improvement is so hard to make.

International Comparison of Student Achievement

Comparisons between countries are often difficult, and at times can be misleading. This is because countries are large entities and often, but not always, the effectiveness of schools varies widely within a single country. In this section we share how the US does on average compared to other countries in mathematics assessments, but we also try to illustrate how student achievement varies within the US.

Two large, well-designed, international studies regularly test how US students compare with students around the world: the Third International Mathematics and Science Study (TIMSS) and the Program for International Student Assessment (PISA). TIMSS studies students' knowledge of mathematics and science about every four years. PISA examines students' knowledge of mathematics, language arts (reading/writing), and science every three years but the focus of the exam alternates between the three topics. Are the scores on these exams worth paying attention to? Do they substantially test knowledge that is important? There is evidence that the answer is yes. For example, the PISA study has followed students in a longitudinal study to see how well the scores on PISA exams predict success in college and in the workforce (Schleicher, 2007). They found that PISA scores were the best predictor of success, better than national exam scores or even teacher ratings on how successful students would be in college and the workforce. Teacher ratings ended up being the poorest of the three indicators and were often biased based on the students race or social background. These international exams do tell us something important about the knowledge of students.

The 1999 TIMSS results show that the US fourth graders do relatively well, scoring above the international average but being outpaced by the top countries. By eighth grade US students had fallen to a level just below the international average, with many more countries scoring higher than the US students (NRC, 2001). The US students fall even further behind by the end of high school. US students in the 12th grade only scored significantly higher than two of the twenty countries that participated in 12th grade testing (Glenn, 2000).

The most recent results show that the US has made some progress from the 1995 and 1999 TIMSS studies. The 2007 TIMSS results show that the US fourth-graders have moved up an average of 11 points from 1995 (518 points) to 2007 (529 points) (Mullis, Martin, & Foy, 2008). The top four countries in 2007 had fourth grade scores of 607, 599, 576, and 568. In that same time span some countries made very large jumps. England went from 484 to 541 and Hong Kong went from 557 to 607. Most countries had scores that stayed fairly constant (within 10 or 20 points from their 1995 scores) but some had large drops, for example, the Czech republic went from 541 to 486.

The US eighth grade scores showed even greater improvement (Mullis, Martin, & Foy, 2008). We increased 16 points from our 1995 score of 492 to 508 in 2007. The top eighth grade scores were 598, 597, 593, and 572. Notice that the 2007 US scores show a drop from 4th to 8th grade. This is also seen in a greater difference between the scores of the top countries and the US average score in 4th grade than in 8th grade. The US students slip further behind the top countries in the four years from fourth to eighth grade.

Although the TIMSS scores show a slight increase, the PISA scores show a decrease from the year 2001, the first year of testing. In 2001 the US score was 493 and the average score for OECD countries was 500.

By 2006 the US score had fallen to 473 and the OECD average was 498. This may seem to conflict the evidence from the TIMSS studies but there are some differences in the tests and administration. The PISA examines 15 year old students where as the TIMSS examines eight graders. The results suggest that we are doing a little better up to eighth grade but we are doing worse with our students as they enter high school.

Fewer countries have participated in the 12th grade portion of the TIMSS studies, and they do not perform the 12th grade study each time. In 1999, twenty-one countries had 12th grade students participate in the study. The US only scored better than two of these countries (Glenn, 2000) and fell well behind the international average by 40 points. This study did not include any of the traditionally top scoring countries of Japan, Hong Kong, Singapore, Taiwan, or Belgium. No new large scale studies comparing 12th graders internationally has been done since 1995, so it is hard to know how the US students currently stand among their international counterparts when leaving high school.

Benchmarking Studies

There are a handful of states whose students are on par with all but the highest achieving countries, at least in eighth grade. For example, some states scored well enough that only six of the 41 countries scored significantly higher on the eighth grade TIMSS mathematics exam. However, the lowest states scored significantly higher than as few as three of the 41 countries. Some districts have also been included in TIMSS and scored as their own country. A consortium of districts in the Chicago suburbs scored well enough on the eighth grade exam that only Singapore, the highest achieving country in the study, scored significantly higher (NRC, 2001).

Comparison of Top Students

A common response to these studies is to attribute the differences to the top achieving countries only educating an elite few whereas in the US we focus on educating all of our students. There were times when this was true. The US was the first major country to invest in education for all citizens. In the 1950's we graduated a greater portion of our citizenry from high school than any other OECD country (Schleicher, 2007). However, this is no longer the case. Many countries have surpassed the US in the proportion of citizens they graduate from high school (it is not that the US graduation rate has dropped, but that the other countries have caught up and passed the US). High achieving countries tend to have an equal (or higher) proportion of test age citizens taking the international tests as the US does (OECD, 2007).

The distribution of scores from international comparisons, such as the TIMSS, is enlightening about where our top students stand compared to other countries. Our average scoring eighth grade students did not make it into the top seventy five percent of scores in the six highest scoring countries in the 1999 TIMSS. Students scoring in the 95th percentile (scoring higher than or equal to 95% of the students) in the US would not make it into the top 25 percent of scores in the three highest achieving countries. How do the low scoring students in the top achieving countries do compared to US students? Students in the 5th percentile of four of the top five high achieving countries would score higher than about 25 percent of the US students.

In an earlier version of the TIMSS studies there was an advanced mathematics test given to students in the top 10-20 percent of students in twelfth grade (or last grade of secondary school as some countries only have eleven grades) (Mullis et al., 1998). The US students tended to be in calculus and college algebra courses. The advanced US students were only able to score higher than one of the 16 countries. The sample also did not include any of the traditionally higher scoring Asian countries. Only the top 25% of US students were able to score higher than the international average. The average US score (442) was at least 100 points lower more than the top two countries, France and the Russian Federation.

Ball and Hill and colleagues developed tests that evaluate this kind of mathematical knowledge and called it Mathematical Knowledge for Teaching (MKT). Elementary school teachers that have higher MKT scores also have students that score higher on standardized achievement exams (Hill, Rowan, & Ball, 2005). High MKT teachers also have more accurate and higher quality mathematics instruction, which of course, could be a central reason why their students tend to learn more (Hill et al., 2007).

The MKT tends to be *deeper* knowledge of mathematics, as opposed to *further* knowledge of mathematics. MKT can be learned, but it is not fully learned in college mathematics courses. MKT lies at the intersection of school mathematics and students learning school mathematics, with knowledge of further mathematics providing the horizon to which students are traveling.

One of the most interesting findings thus far about MKT is that there is evidence that this kind of mathematical knowledge is a different kind of mathematical knowledge that is used by mathematicians or others in mathematically intensive fields. There are some problems on MKT tests that experienced elementary school teachers understand better than mathematicians (Sleep et al., 2005). This is not to discount the help mathematicians can offer in helping to train teachers (some mathematicians were involved in creating the MKT exams as well), but this suggests that there is a piece of the mathematical terrain that is better understood by experienced mathematics teachers, even at the elementary school level.

Mathematics teachers in the US, particularly elementary school teachers, tend to have a weaker knowledge of mathematics than mathematics teachers in high-achieving foreign countries. Some high achieving countries achieve a high level of mathematical knowledge by having elementary school teachers specialize in one or two subjects (Li, 2008; Ma, 1999). This frees up time for teachers to develop the deeper knowledge of mathematics that allow them to be effective in the classroom. College courses and professional development opportunities spread across many topics [reading, writing, mathematics, social studies (which includes history, civics, political science, geography, economics, etc.), physical science (which includes biology, earth science, chemistry, physics, etc.), art, music, physical education, health, and computers] can not effectively develop the deep knowledge and the further knowledge needed to develop adequate knowledge in teachers to teach well and inspire students in each of these subjects.

Some professors and some secondary teachers spend their whole careers studying the learning and teaching of one subject. It seems unfair to expect US elementary teachers to know each of the many subjects they are required to teach as well as those who focus on one discipline. This problem is exacerbated in the US because we expect our teachers to teach more (see the earlier section on curriculum) and thus need to know more. Remember too that US teachers tend to be those in the lower third of the academic distribution compared to high achieving countries which draw teachers from the highest third (and in some cases the highest ten percent) of the academic distribution.

Enrollment Growth vs. Employee Growth

Over the last century the ratio of the number of kids enrolled in public schools and the number of teachers has dropped almost continually. But the drop has been most dramatic in the latter part of the century. The public school enrollment has doubled since about 1946 (Snyder et al., 2009). (For a quick view of the data see http://social.jrank.org/pages/1025/Teachers-Teaching-Data-Presentation.html.) However, the number of teachers has almost quadrupled in that same time period, from 831,000 to about 3,200,000 (cite). This has allowed teacher-pupil ratios to be almost cut in half from the late 1940's.

The number of non-instruction employees, such as administrators and support staff have grown even faster then the number of teachers. This is one reason educational spending has been growing so rapidly. Figure 1 shows the number of enrolled students and number of employees (instruction and non-instruction) since 1972. As the figure shows, the growth of public school employment has nearly quadrupled while student enrollment is only slightly higher than 1972 levels.

For example, in Japan, it is not uncommon for beginning teachers to have a written out, detailed lesson plan on their desk for every lesson they teach. Teachers in high achieving countries are usually allowed more time to prepare their lessons each day, but often are asked to do other administrative tasks that are not required by teachers in US schools. They tend to share office space with other teachers and collaborate on lessons.

This may seem paradoxical that many of the brightest students in high achieving countries want to be teachers even though the work days are longer, the expectations are high, and the work load is greater than for US teachers, often with mediocre pay. (Although this last point about pay varies greatly among high achieving countries with some teachers being paid much higher than the median income for that country). But the environment tends to be nicer to work in with administrative and collegial support, good student teacher relationships and a relaxed atmosphere where learning is valued.

Teacher's Knowledge of Mathematics

Teachers need to know mathematics well to teach mathematics effectively. Several international comparisons, although some of them are small, show that US teachers, particularly elementary school teachers do posses lower mathematical knowledge of mathematics than teachers in high achieving countries (Ma, 1999; Ball et al., 2001; Hill et al., 2007). Part of this is simply due to the fact that high achieving countries are producing more knowledgeable citizens and some of those citizens become teachers. Remember that high achieving countries draw teachers from the upper third of their academic distribution while the US draws from the lower third (Schleicher, 2007). This exaggerates the difference in teachers' mathematical knowledge from high achieving countries to the US.

When someone says that teachers need to know more mathematics, there are two basic ways of interpreting what "more mathematics" means. This could mean that teachers should learn more advanced mathematics, usually interpreted and implemented in the form of college level mathematics often in college mathematics courses. Another way to think about knowing "more mathematics" is to understand the mathematics that students are learning better. One can be thought of as going *further* in mathematics and the other can be thought of going *deeper* into mathematics. Although both kinds of knowledge are needed to a certain extent, researchers have explored which kinds of knowledge have a larger impact on student learning.

Teachers that take more college mathematics tend to have slight increases in student achievement, *but only if they have taken very few college mathematics courses* (Beagle, 1979; Monk, 1994). After about five college mathematics courses taking more college mathematics courses does not significantly help teachers improve their effectiveness. Thus requiring all secondary mathematics teachers to get a masters degree in mathematics would be a policy that current evidence suggest would lead to little or no increase in student learning.

In the same study that showed that there are threshold effects of advanced mathematics classes, researchers found a positive effect for the number of mathematics methods courses that teachers had taken (Monk, 1994). This gives a key on what knowledge of mathematics may be most beneficial to teachers: knowledge of mathematics closely connected to the mathematics that they are (or will be) teaching. Mathematics methods classes offer opportunities for teachers to learn primary and secondary mathematics better and insight into how students think about and learn mathematics.

More recently researchers have explored how detailed knowledge of elementary mathematics helps effective teachers. Deborah Ball and Heather Hill and their colleagues at the University of Michigan looked closely at the mathematical work used by teachers as they teach mathematics to students. Some teachers actually do a lot of mathematics while preparing their lesson and teaching students mathematics. Knowing the advantages and disadvantages of using particular examples, evaluating student strategies, checking student work, evaluating the correctness of textbooks, choosing mathematically correct representations, forming responses to student questions, as well as other tasks, all require solving mathematical problems at times or drawing upon knowledge of mathematics.

Two studies, seven years apart, found that by the fifth grade the highest achieving US mathematics classroom in the sample was lower than the lowest achieving mathematics classroom in the sample from Taiwan or Japan. At first grade the highest achieving US classroom scored at the average of Japanese classrooms (Stevenson & Stigler, 1992). In one of the studies mentioned above only one US student made it in the top 100 scores, eleven were from Taiwan and eighty eight were from Japan (Stevenson & Stigler, 1992).

US students fall further behind the harder and more complex the task. A study of 13 year olds in Korea and the US found that when asked basic facts about science the US students did fairly well scoring 96 percent correct compared to the 100 percent correct for Korean students. But as the questions got harder and application of principles and analysis was involved, US students did much worse than the Korean students with scores of 78 percent, 42 percent, and 12 percent correct on problems where Korean students received 93 percent, 73 percent, and 33 percent correct respectively (Sowell, 1993).

Physical Resources

It may seem odd for many Americans to learn that the US tends to have nicer schools than almost every large civilized country. The typical school building and school grounds in the US would be the envy of schools around the world. US schools also tend to have more computers per child, many more books, and much newer textbooks. Most elementary schools in Japan or China, for example, have no school library, and no cafeteria. Many do not have a gymnasium. The classrooms are not nearly as spacious because they fit 38 to 50 students in a classroom. There is very little room in a classroom for a library corner, displays of animals and plants, or work areas like those in typical US classrooms. Asian school buildings, particularly in China, are very similar to US schools built fifty or sixty years ago (Stevenson & Stigler, 1992). Many of these school buildings do not even have heating.

It would seem that US students and teachers are lucky to have such nice buildings to work in, but they have not seemed to translate into greater learning. Asian elementary schools are much more efficient at helping students succeed in school than US elementary schools. One need only look at the first year of school to see this difference. There is an emphasis in America to read to young kids before they enter school and to work with kids to know their alphabet and sounds. No such emphasis exists in the East Asian countries of Japan, Taiwan, and China. Because of parents working with their young kids, American students enter first grade at an average reading level equal to or higher than Japanese or Taiwanese students. However, by the end of first grade American students are already behind their East Asian peers in reading. The average American student falls further behind each year of schooling (Stevenson & Stigler, 1992).

Do these results come at a price?

Many Americans could probably share rumors about the extensive work required by Asian students to succeed in school: long hours at school, many more hours of homework, evening cram schools, and little time to play or enjoy themselves. Although some of these may be true during secondary education, there is little evidence that this is the case in elementary school. School days are longer in East Asian schools, but much more of the day is spent on non-academic material including music, crafts, sports, clubs, etc. In elementary school East Asian students only spend about an hour more per day on homework than US elementary school students. Asian elementary school students actually spent more time watching television than US students. *The Learning Gap* by Harold Stevenson and James Stigler (1992) is a classic book that compares elementary schools, teachers, and students in the US, Japan, Taiwan, and China and debunks many of the myths rampant in the US about Asian schooling. Many of the high achieving European countries share many of the principles used in Asian schools (OECD, 2006).

Do these results come at an economic price? The answer is **yes** if you consider how much lower level skills impact our national economy or an individual's opportunity to succeed in college or the workforce. Low levels of achievement come at a great price to the US and its citizens. If the US students had been performing near the top of their international competitors the US GDP would increase an estimated \$1.3 to \$2.3 trillion dollars, or 9%-16% increase in GDP (McKinsey & Company, 2009). More on the economic impact of low scores is discussed later in the report. At the individual level the impact is even greater. A student that receives a bachelors degree earns almost 70% more than a student that only has a high school diploma (US Department of Education, 2008) The gap is widening. For example, in 1980 the difference in pay was only 14.5% more for a bachelor's degree.

International and National Comparison of Non-student Achievement Variables

Many educational policies focusing on improving outcomes focus on changing *teachers* rather than *teaching*. The policies focused on teachers aim to recruit and retain better, more capable individuals to teach our kids. These policies try to change who is standing in front of the classroom. Although there are issues about recruiting and retaining teachers, which we discuss later, we argue that it is more important to focus on teaching, instruction, and on what students experience in school than on changing who is running the classroom. The quality of the mathematics instruction is an area where we fall far below the high-achieving countries. Moreover, the instructional quality is largely under the control of schools and teachers. Reasons for our low test scores attributed to students, parents, society, or other non-school factors can hold little weight if the quality of instruction in school is mediocre.

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Mathematical Content

As part of the TIMSS 1999 study (Hiebert et al., 2003) more than 50 eighth grade mathematics lessons were video taped in each of seven countries or regions: the Czech republic, Switzerland, the Netherlands, Australia, Hong Kong (SAR), Japan, and the United States. The US was the lowest achieving of the seven countries. The US also had the lowest quality mathematics instruction. Analysis of these eighth grade mathematics lessons showed that it was not just one feature of US mathematics instruction, but a constellation of characteristics that all contribute to poor instruction. Hiebert et al. (2005) argue that compared to other countries, US instruction can be characterized as "frequent reviews of relatively unchallenging, procedurally oriented mathematics during lessons that are unnecessarily fragmented" (p. 116-117). Their paper shows the rich and stark differences that we do not have time to go into in this report. This study adds to the evidence that US students can learn more mathematics, and can learn more mathematics better, than they are currently learning. But it will take better instruction for this to happen. The mathematical basics that we currently expect our students to learn are limited and shallow. The low-level work that we require does not generally value student thinking nor does it develop habits of mind that match those of similar aged students in competing countries.

A central feature, and perhaps *the* central feature, to high-quality instruction, both in the US and abroad is this: students must do some intellectual work with important mathematics. We emphasize this feature because of the strength of this finding from empirical research. In a review of over three hundred documented studies or cases where students were learning mathematics *with understanding* (not just being able to perform procedures and recall basic facts) there were only two characteristics that were common among all of these cases. The first one is the finding that we illustrate here, that students must be doing some intellectual work, or some serious thinking, about important mathematics. The authors that pointed this out used the words that students have to *struggle* with the mathematics. But they clarify what they mean by the word struggle:

> We use the word *struggle* to mean that students expend effort to make sense of mathematics, to figure something out that is not immediately apparent. We do *not* use *struggle* to mean needless frustration or extreme levels of challenge created by nonsensical or overly difficult problems. We do not mean the feelings of despair that some students can experience when little of the material makes sense. The struggle we have in mind comes from solving problems that are within reach and grappling with key mathematical ideas that are comprehendible but not yet well formed. By struggling with important mathematics we mean the opposite of simply being presented information to be memorized or being asked only to practice what had been demonstrated. (Hiebert & Grouws, 2007, p. 387-388)

A clear difference between high achieving countries and the US is where and how we recruit teachers. High achieving countries draw teachers from the highest third of the academic students. The teacher education programs in these countries are very competitive and draw top students. In the US our teachers tend to come from the lowest third of the academic distribution (PISA, 2007). Teacher education programs in the US are easy to get into, are considered easy majors, and draw many students that could not succeed in other majors, particularly the hard sciences (Sowell, 1993). One study found that only seven percent high school seniors with SAT scores in the highest 20 percent consider going into the teaching profession, 13 percent of the next quintile consider teaching. Compare this to about half of all students in the lowest two quintiles consider going into education (Sowell, 1993). Education majors also tend to have the lowest GRE scores compared to other majors (Educational Testing Service, 2007).

Half of all beginning school teachers leave the profession within five years (Parsad et al., 2001). The reasons vary from taking time off to have a family, going to graduate school, moving to an administrative position, changing professions, or being laid off, etc. The most common reports in a national survey found that the main reason for turnover and attrition were inadequate administrative support (38 percent) and workplace conditions (32 percent) (Parsad et al., 2001). Unfortunately for our students, the more academically capable teachers tend to be among the ones that leave the classroom more readily. For example, of those that score in the top twenty percent on the SAT that actually teach, 85 percent of them leave after brief careers (Sowell, 1993). Among mathematics education and science education majors in junior high and high school scores tend to be higher than among other education majors. Elementary school teachers tend to get the lowest scores (Sowell, 1993).

One popular suggestion to improve teacher recruitment, especially in Utah, is to increase teacher salaries. This would hypothetically draw more capable people into teaching. This may help, however, salary is not the major reason people go into teaching or leave the teaching profession (Schleicher, 2007), but it is unknown where salary falls for people who do not choose teaching in the first place. There is something about the environment and work of teaching in US schools that is very taxing on teachers that makes many not even consider teaching in the first place, and leave quickly if they do teach.

Some countries do use high salaries to attract the best teachers. For example, Korean teachers are paid 2.3 times the countries per capita GDP. In the US the average teacher is paid about 1.2 times GDP. The national average pay of public school teachers was \$49,109 in 2006. The rate does vary by state though and goes from about .8 per capita GDP to 1.4 per capita GDP. Utah teachers are paid about 1.1 times per capita GDP (average of \$40,316 in 2006). But Finland is an example of a country that has gone a different path. Their teachers are paid only average for OECD countries, but they created another set of incentives where they focused on making it a knowledge rich profession where teachers have a lot of control and are held accountable for the quality of their work (OECD, 2007). There is extensive training and sharing of ideas. They have succeeded in creating a profession of teaching that is more like that of a professor. The position comes with status and respect. In Finland this change has produced a surplus of teachers, with nine applicants per position, and has helped them reach the pinnacle of international achievement. Finland was the highest achieving country in the 2006 PISA study (OECD, 2007).

Teach for America (TFA) has had a lot of success recruiting top achieving college grads in fields outside education to staff inner city schools. They are trained in classroom discipline techniques as well as other practical teaching skills. These TFA teachers pick up the work of teaching quickly, and partially because of their strong academic background, inspire students to do challenging academic work (Rhee, 2009).

We have seen that many US teachers leave the profession. High achieving countries do not share this same trait. Teachers go into teaching and view it as a professional career. The environment in these countries tends to match more closely to that of a university professor rather than US K-12 teachers. There are high expectations that accompany these teaching posts. Their lessons are to be well thought out and polished.

Curriculum Breadth and Depth

Schools in the US teach many more topics than high achieving countries, especially in elementary school. For example, one study found US fourth grade teachers "cover" about twice as many topics as fourth grade teachers in high achieving countries (32 in the US, an average of 18 in other countries) (Schmidt et al., 1999). Exactly how many more topics the US covers depends on how the word *topic* is defined and at what grade level the study focuses on. Another study found that while students in Japan covered four topics in first grade and four in second grade, US students covered eleven in first grade and twelve in second grade. Even more dramatic are the results of a recent international study that showed that the five top high achieving countries only teach about 15% of what is in the US curriculum (Rhee, 2009). Some differences are expected, but this large of a difference is extreme.

This situation of trying to help students learn with a curriculum that is "a mile wide and an inch deep" leads to student understanding that is weak. There is not enough opportunity for typical students to learn mathematics really well. High achieving countries tend to focus on a few key topics each year, and help the students learn it well so that time does not need to be used in later grades reviewing. Curriculum studies from the TIMSS studies show that mathematics topics are in the US curriculum for longer than all high achieving countries, and longer than all but a few countries in the study (Schmidt, 1999). High achieving countries actually spend very little class time reviewing material (the Czech Republic is a counter example to this trend which uses review in a novel way) (Hiebert et al., 2003). Because US students do not tend to learn the mathematics well at any one grade many of the topics are covered again in the next grade, which cuts out valuable time for learning new material (and remember that US students already spend less time in school than students in high achieving countries). This creates a cycle that contributes to US students falling further behind each year in school.

Administration

In general, high achieving countries have schools and schools systems with far fewer administrative positions than US schools. One strategy that is used in Japan, as well as other countries, is that few school positions are full-time administrative positions. The administrative responsibilities are taken on by the teachers. Some of the teachers are given reduced loads to fulfill administrative duties. This helps in a few ways. First, it still allows the best teachers to continue teaching. In the US system many of the best teachers soon get promoted to administrative positions where they are replaced with younger, less qualified candidates. Second, money spent on full time administrators can be given to teachers for performing the administrative responsibilities. This improves teacher pay, although it also increases their work load. Third, reduced load administrative positions provide a way for teachers to advance in their field, but still remain teachers. In the US a teacher of three years is largely the same as a teacher of thirty years. The differences in responsibilities and status are small compared to the teacher advancement system in Japan and other countries. In Japan teachers can advance levels in their school and in their field much like engineers or analysts advance from being level 1 analysts to senior research analysts.

In short, high achieving countries minimize administrative work, and when it is necessary they delegate what they can to teachers by giving them reduced loads. In this way they can keep the best teachers in the classroom and capitalize on their knowledge and experience. Later in the report we discuss how states vary widely in the number of non-instruction employees.

Recruiting and Keeping Teachers

Before discussing this topic I want to be clear that in general US and Utah teachers are very capable and hard working. They face many challenges and work hard to do the best they can given the challenges. This section is in no way meant to degrade the important and valuable work done by our teachers.

It is important to emphasize that there has not been one documented case that we are aware of where there is strong evidence that all students were learning mathematics *with understanding* without students engaging in struggling with important mathematics. All evidence points to this characteristic being a necessary factor for effective instruction if a goal of instruction is to help develop mathematical understanding within students. So if students are not being supported in struggling with important mathematics during a series of lessons, then all evidence suggests that students, as a whole, will not be building mathematical understanding.

One of the most dramatic findings of the most recent TIMSS video studies is the high association between countries where students are required to "struggle" in class and the average achievement of that country (Hiebert et al. 2003). As part of the video coding process lessons were judged on the type of mathematics students did during the lesson. Students were given problems where they were expected to respond in one of four different ways: giving results only, using procedures, stating concepts, and making connections. The amount that students had to "struggle" (captured in the *making connection* codes) were highest in the high achieving countries and lowest in the low achieving countries, with less than one percent of US lessons requiring students to do high-level, making connections work.

Some US teachers capture the essence of this idea in statements such as: "Never do for a student what he or she can do for themselves" or "When we do for others what they can do for themselves, we generally weaken rather than strengthen them." But too few lessons in US classrooms support students in working on and learning mathematics in such a way that challenges them to the level that is common among high achieving countries. This is especially true as US students progress in school, when students are becoming more capable and can deepen their intellectual skills and mathematical understanding.

Teaching Methods

One clear finding from national and international studies is that effective instruction can be based on different methods. Debates have raged about how to best teach mathematics in school. One key finding from the mathematics literature is that a variety of approaches can be effective and that instructional methods are a weak indicator of effectiveness (Weiss, 2003). Internationally Japan and Hong Kong SAR are two of the highest achieving regions, however, Japanese mathematics instruction looks very different from instruction in Hong Kong SAR (Hiebert et al., 2003). In Japan, teachers tend to spend a lot of time exploring a few problems in depth with a substantive amount of student-student interaction (often in groups), whereas in Hong Kong SAR teachers give an interactive lecture and engages the students in solving many problems. The instruction in Hong Kong SAR looks, on the surface, like typical US instruction, but there are some important differences that make a large difference: coherent lessons that build naturally from students' current understanding, high expectations and support to master challenging content, and of course, opportunity to do intellectual work on important mathematics.

Studies within the US have also shown that effective instruction can be based on different methods. The Inside the Classroom (Weiss, 2003) study rated the quality of classroom instruction and found that there was high-quality instruction in both "traditional" and "reform" classrooms and low-quality classrooms in both kinds as well. No single method seemed to be the key for making an effective classroom. Although effective instruction can be accomplished using many methods that does not mean that any method is effective. Some practices are nearly always detrimental or ineffective.

Use of Class Time

Classroom time where teachers are spent actively teaching is much more effective than students spending it in unguided seatwork or homework. This finding may at first seem at odds with the finding that students need to do intellectual work on important mathematics. Isn't seat work the time where students struggle with the mathematics? It can be, but students left to themselves in unguided seatwork do not receive much support in mathematical thinking. This is not to say that students should not do seatwork or practice problems, they should, but large portions of the class spent with unguided seatwork tends to be a less effective use of class time.

A recent international study found that an hour of class time is worth three to four hours of homework in the amount that students learn (OECD, 2007). This is evidence that students tend to learn more when the teacher is active in helping students to learn mathematics. Another study had teachers experiment with the amount of time they spent on the *developmental portion* of the lesson (Good, Grouws, and Ebmeier, 1983). The developmental portion of the lesson is the portion of the lesson where teachers were explaining and developing the mathematics as opposed to having students doing seat work. Some teachers spent 25 percent of the time on the developmental portion of the lesson while other teachers spent 50 percent of the time or 75 percent of the time. The remainder of the time was spent in seatwork. Teachers that spent 75 percent of the time developing the mathematics had the highest scoring students followed by the teachers that spent 50 percent and then 25 percent. Spending more time developing the mathematics allowed students to make more connections and gain a better understanding even though it decreased the amount of time in class practicing problems by themselves.

Another common attribute of high achieving countries is that they spend class time learning *new* material and very little class time reviewing previously taught material. (The Czech republic is a counter example where they use daily review in a novel way to hold students accountable on the material covered the previous day). The US, as found in the study by Hiebert et al. (2005), found that the US spends a large portion of class time reviewing material. This diminishes students opportunity-to-learn.

Mathematics Instructional Time

How much mathematics instruction do elementary school students receive everyday? On average it is about forty-five minutes, but some students receive much more and some students receive much less. One reason some students receive much less is because some teachers do not teach math as often as others. In a typical elementary school from a national sample about one-sixth of the teachers missed ten days or fewer of mathematics instruction while one-sixth of the teachers in the same school failed to teach mathematics on more than 43 days (Phelps, Corey, Ball, Demonte, & Harrison, in preparation). That is almost a quarter of the school year! The teachers that were teaching the least were only teaching mathematics about every other day.

When mathematics is taught teachers spend different amounts of time on a mathematics lesson. Because of the large differences in both the frequency of instruction and the duration of instruction, by the end of the year the lowest fifteen percent of students receive about half as many hours of instruction as the top fifteen percent. The top fifteen percent average just over an hour a day in mathematics. The lowest students, the lowest five percent, only receive an average of about twenty minutes of mathematics instruction per school day. These statistics don't even include the instruction missed due to student or teacher absenteeism.

US students are already in school fewer days than their foreign counterparts (often 40-60 days fewer a year) so that by the sixth grade students in foreign countries have had one to two more years of schooling than US students (Stevenson & Stigler, 1992). If US students are to compete with their international counterparts, then making sure students receive one hour of high-quality mathematics instruction could be a simple first step to take.

Class Size

Class size in the US is below the international average in fourth grade (Mullis et al., 2008). The average fourth grade class is 23 while the international average is 26. The US even drops further in class size by eighth grade with the US average class size 24 and the international average 29. The highest achieving countries, both in Asia as well as Europe, all tended to have larger average class sizes than the US, and in some cases much larger class sizes. The higher achieving countries seem to agree that if small class size is important it is much more important in the lower grades than the middle or high school levels.

The idea of raising class sizes would abhor many US parents. But great leverage can be gained in raising class sizes that can greatly improve student learning if the resources that are freed up are used effectively. High achieving countries use large classes to make sure the best teachers are teaching the largest number of students. Keeping class sizes small demands recruiting more teachers which usually means staffing schools with teachers with less capable, less qualified teachers. Low class sizes demand a lot of resources in the form of more schools, more administrators, more staff, and more teachers which depletes resources that could be used in other ways such as teacher salary, teacher training, and teacher instructional support. At least some high achieving countries with very large class sizes help teachers know what instructional strategies and classroom techniques work well for larger classes and how to structure activities to check on individual students (Stephenson & Stigler, 1992). This is not something that US teachers are generally taught in teacher training programs or in professional development activities.

Of course there are documented benefits to low class sizes, however Hanushek(1999) points out in his review of hundreds of class size studies that most of the results show no significant effect and that the number of positive effects of lower class sizes are almost evened out by the number of studies that showed significant negative effects. Where there are significant effects they tend to show up only in very small class sizes (for example Odden(1990) found positive results at a class size of 15) or dramatic reductions in class size. For example, a reduction of ten students in a class only produces an effect size of about .11, which is smaller and less cost effective than other interventions. The aggregate data both from the US and international studies suggest that lowering class size is not an effective or cost effective strategy for improvement. Class size has continually dropped in the US at least five decades but there has been no associated dramatic increase in student achievement.

Curricular Control

Practically all high achieving countries have a detailed national curriculum that specifies what students should learn in each grade (Schleicher, 2007). Of course some low achieving countries also had a national curriculum. A national curriculum makes it easier to coordinate improvement efforts, drives down the cost of textbooks, eliminates many of the problems associated with student mobility, and helps in other ways as well (Hirsch, 1999). In no other high achieving country that we know of do teachers, schools, or districts have control over what content is taught to students to the extent that happens in the US. Teachers and schools in high achieving countries do have control over how it is taught, but are held accountable, in various ways, for the results.

US students in the same school and in the same grade often receive very different amounts of mathematics instruction and cover different mathematical content, let alone students in different schools in different states (Porter, 1989; Rowan, Harrison, & Hayes, 2004). It is not uncommon for one teacher to spend twice as much time, or more, teaching mathematics than another teacher in the same school. The amount of time one teachers spends on topics varies widely from other teachers. Just because a topic is taught in one class does not mean that it is covered in another class.

There are high achieving districts in the US that are on par with the highest achieving countries, so a national curriculum is not necessary for marked local improvement. More research is needed to see the extent that these high achieving districts control the mathematics content taught to students at each level.

Some readers may be asking, don't states have a set of mathematics standards that sets out what students should be learning at each level? Yes. But studies show that many teachers do not use them to plan out their mathematics instruction for the year, especially in elementary school (Darling-Hammond, 1990). What do teachers teach? Most teachers teach what is in their textbook, starting at the beginning and going lockstep through the sections until the end of the year, with only slight variation, whether the textbook sections are what the students need or not (Ball, 1990; Ball & Feiman-Nemser, 1988; Schmidt et al., 2001). High achieving countries have various ways of maintaining adequate quality control on content but their methods go beyond the scope of this report.