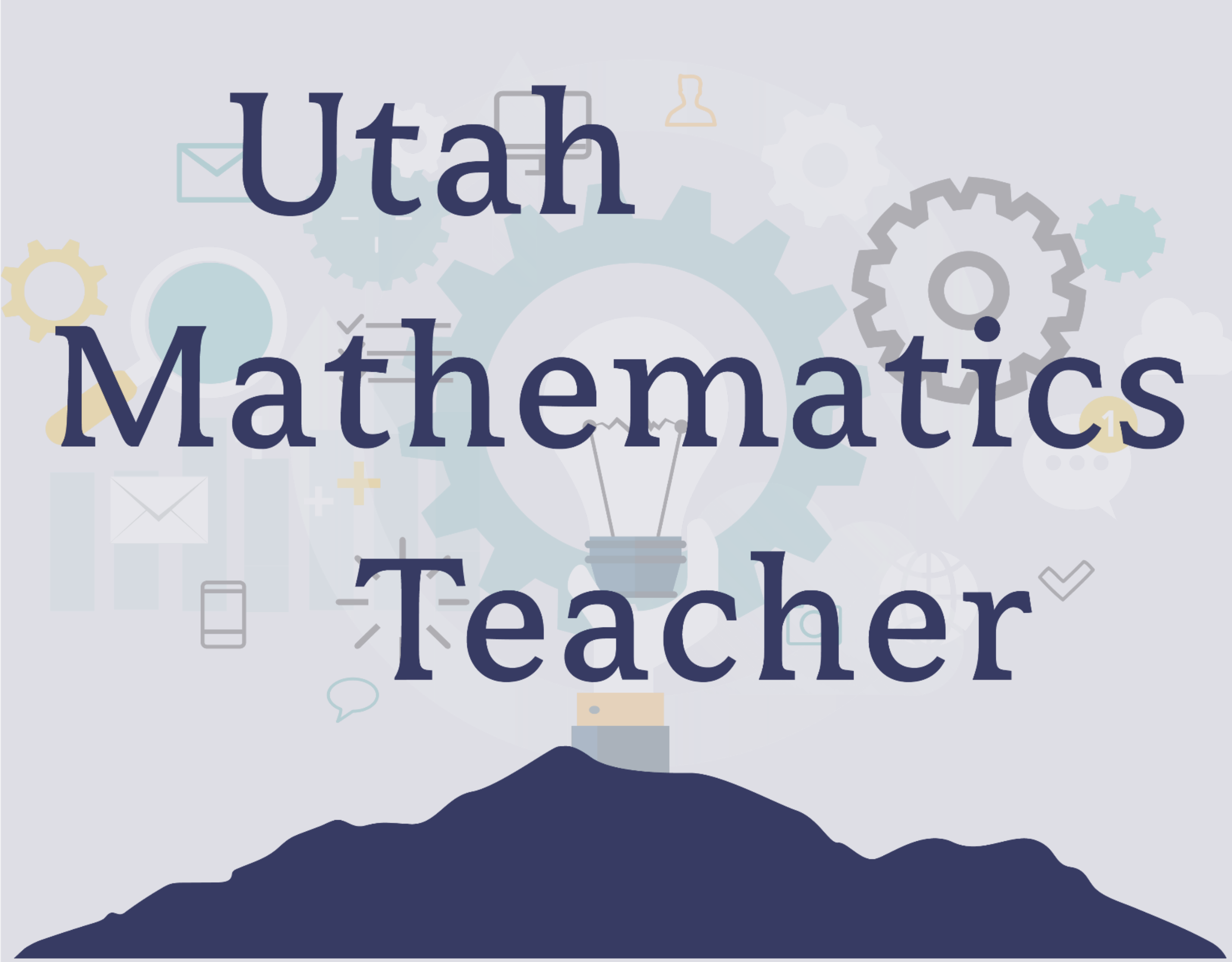
Utah Council of Teachers of Mathematics

Utah Mathematics Teacher



Volume 13

Fall 2020

"A Year of Innovation"

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CALL FOR ARTICLES

The Utah Mathematics Teacher seeks articles on issues of interest to mathematics educators, especially K-12 classroom teachers in Utah. All are encouraged to contribute articles and opinions for any section of the journal. Manuscripts, including tables and figures, should be typed in Microsoft Word and submitted electronically as an e-mail attachment to Danielle Divis at danielledivis21@gmail.com. A cover letter containing author's name, address, affiliations, phone, e-mail address and the article's intended audience should be included.

UTAH MATHEMATICS TEACHER

Volume 13, Fall 2020

"A Year of Innovation"

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UCTM President Elect's Message

Andrew Glaze, President-Elect

I am what one might call a guitar hobbyist. To say that I have mastered the instrument would be one of the overstatements of the century, but after two decades I continue to learn, practice, and play. Many years ago, I observed a street musician playing flamenco style guitar on what I would have considered an unplayable instrument. In addition to various holes and general dilapidation, the guitar was missing a string and the other five looked as if they were a whisker width from the same fate. I watched in a bit of awe and amazement as the aged performer played a spectacular arrangement of music that did not sound as if it originated from the guitar in his hands. Prior to seeing him play I would have thought the instrument incapable of making music. That street performer was innovative. He was using an old and tired instrument in an awe-inspiring fashion.

Recently I observed a teacher engaged in a remote lesson. To be truthful, I thought the curriculum was a bit constrictive and the technology too cumbersome for the monumental task of building student understanding and engagement in an online lesson. I wondered how she was going to pull off this lesson. What I saw after the class began left me both inspired and more than a little awed. With the use of some solid pedagogical practices and an attention to student needs, this teacher turned an otherwise uninspiring set of circumstances into a work of educational art. It was impressive to say the least. Students were engaged with the mathematics. Formative assessments were artfully built into the lesson, the teacher was responsive to the students' needs, and the technology was used to leverage evidence of student work and understanding. In my eyes, this teacher was working with an online mathematics class in much the same way that the street musician was working with his guitar.

Most of us probably want better technology, new or better curriculum, and students in our classrooms. May I suggest that we don't need something new as much as we need innovative uses of what we already have. We're in this boat together, so jump in with both feet and be sure to share it so that we can innovate together!

Letter from the Editors

Christine Walker, Utah Valley University
Danielle Divis, RSL Academy High School

Every new school year begins with a challenge! However, this year has been unprecedented given the utmost concern of the public health and safety of our schools. As communities and districts across the state of Utah have grappled with safely re-opening, K-12 school administrators have devised plans to help protect students, teachers, and staff to slow the spread of COVID-19.

What has been your challenges and successes so far? As we have interviewed teachers we have surmised that the single biggest challenge for you right now is making sure that students do not fall too far behind. In this journal the articles will address and show effective practices for mathematics teaching and learning, and will provide helpful ideas and suggestions to address the challenges that teachers face teaching in-person, hybrid, and online. As such, our UCTM theme this year is "A Year of Innovation."

We open the journal with a message from the NCTM President challenging teachers to participate in NCTM's *100 Days of Professional Learning* "and continue to see themselves as lifelong learners." It is a unique opportunity to gather together as professionals remotely and to learn from one another regarding "mathematical learning experiences for each and every student."

https://www.nctm.org/News-and-Calendar/Messages-from-the-President/Archive/Trena-Wilkerson/Celebrating-Professional-Learning/

We turn our attention to our *featured* articles. We begin with an article, *The Critical – Identity Construction – Empowerment (Crit-IC-E) Conceptual Framework*, with "one goal of mathematics educators…to empower students to be confident, capable, crucial users and creators of mathematics." The author assets that "critical mathematics educators employ critical theory as the framework for their work."

The next featured article, *A Different Approach to Trigonometry*, compares and contrasts the right triangle method versus the unit circle and shows that "trigonometry can be taught without making students memorize the unit circle" by shifting the focus from the "unit circle to the graphs of basic trigonometric functions."

In our next featured article the author writes, "I was recently inspired by an email I received from a 6th-grade teacher regarding her desire to meet and discuss ways to restructure her 6th-grade math Response to Intervention (RTI) instruction. This portion of instruction is in addition to the whole group instruction, and meant to (a) intervene for students who have not yet grasped the content, and (b) enrich the knowledge of students who already demonstrated proficiency." The article, *A Math RTI Structure to Support Equity and Learning*, introduces the reader to "an effective mathematics Response to Intervention (RTI) structure…to effectively help all students feel valued in their mathematics learning and to show mathematics learning growth."

Our final featured article, *The Lost Voices in Mathematics Teaching*, is geared towards Higher Education. Yet there are lessons that can be learned and applied to K-12 in addition to higher education as the author explores the idea of discovery learning as a means for learners to "acquire knowledge on his/her own facilitated by teachers' tasks and questions."

We hope you enjoy this journal as much as we enjoyed collecting, reading, reviewing, and discussing the articles with the review committee. In addition, as always, please consider submitting your own articles, or serving as a reviewer for future journal articles.

Note: Any mistakes are the sole responsibility of the editor and will be remedied in the online journal. Please send corrections to Danielledivis21@gmail.com or Christine.walker@uvu.edu.

The Critical - Identity Construction-Empowerment (Crit-IC-E) Conceptual Framework

Carrie Bala, Utah State University

One goal of mathematics educators is to empower students to be confident, capable, critical users and creators of mathematics. As high school students encounter mathematics, this goal is not always made obvious to them. In the NCTM publication Catalyzing Change in High School Mathematics, the authors call for teachers to make clear the purpose of high school mathematics: “High school mathematics empowers students to – expand professional opportunity; understand and critique the world; and experience wonder, joy, and beauty” (NCTM, 2018, p.9). These purposes align with the perspective of critical mathematics educators.

Critical Mathematics

Critical mathematics educators employ critical theory as the framework for their work. With contributions from Marx, Kant, Freud, and Habermas, critical theory developed in the Frankfurt school in the 1920’s. “Critical theory tries to understand why the social world is the way it is and, more important, through a process of critique, strives to know how it should be” (Ewert, 1991, p. 346). Thus, critical theory emphasizes both critique of oppressive structures and action towards emancipation from those structures and an ideal society.

Paulo Freire is credited with the inaugural application of critical theory to the field of education. In his literacy efforts with Brazilian agricultural works, Freire emphasized the importance of developing critical consciousness through education based on the meaningful ideas and experiences of the students (Frankenstein, 1983). Freire utilized problem-posing education, believing that "students, as they are increasingly posed with problems relating to themselves in the world and with the world, will feel increasingly challenged and obliged to respond to that challenge" (Freire, 1978/1972, p. 70). In keeping with critical theorists, Freire highlighted the need for both critique and action, namely, reading and writing the world.

Giroux describes the role of the critical educator through this critical Freirean lens as "mapping the contours of oppression through criticism, a process, which entails both a language of critique and hope" (Leonardo, 2004, p. 16). One such critical educator that centered his work in mathematics education is Eric Gutstein. Gutstein (2006) frames his critical mathematics research in terms of teaching mathematics for social justice. Through his experience teaching mathematics for social justice in a junior high school in Chicago, Gutstein structures social justice pedagogical goals and mathematical pedagogical goals to aid in the development of critical mathematics students. These pedagogical goals aim to assist students in building critical consciousness, reading the world, and empowering students to take action, writing the world.

Gutstein's Mathematics for Social Justice

Gutstein's (2006) social justice pedagogical goals include the Freirean emphasis on praxis. The goal *reading the world with mathematics* means to employ mathematics to develop an understanding of the inequalities of power and resources that exist both in the personal life and the broader community. *Writing the world with mathematics* encompasses social agency meant to improve the local and wider communities. The goal *developing positive cultural and social identities* intends to express strong ties to the home language and culture, adaptability to succeed in the dominant culture, and the self-confidence to act on a sense of social agency.

Similarly, Gutstein's (2006) mathematical pedagogical goals emphasize a combination of reflection and participation. The goal *reading the mathematical world* means to develop the mathematical power to make specific social transformations based in firm conceptual understanding. *Traditional mathematics success* implies access to advanced classes and opportunities, as well as achievement on standardized tests, high school and college coursework. The goal of *changing one's orientation to mathematics* refers to viewing mathematics as a flexible, connected, dynamic, and useful field, rather than a set of disconnected procedures and skills.

As Skovsmose (2016) reminds, "ideas of critical mathematics education have been expressed through general notions like empowerment, social justice, and autonomy" (p. 2). Through a critical mathematics lens, Gutstein (2006) presents social justice and mathematics goals which highlight the role of participation in classroom and community experiences in developing critical mathematics citizens. The set of goals incorporate content learning, positive identity construction, and student empowerment.

Framing Identity Construction

The Content Learning Identity Construction (CLIC) framework for learning, developed by Varelas et al. (2012), sheds light on the connection between mathematics learning, identity construction, and empowerment. According to this theory, learning mathematics is a process necessarily consisting of both mathematical content learning and identity construction. Varelas et al. define learning mathematics as "the intertwining of constructing knowledge with others *and* constructing the self as a learner" (p. 324). The authors further outline the importance of constructing three identities: disciplinary identity, racial identity, and academic identity. Disciplinary identity refers to the positions and characteristics of one who does mathematics professionally. Varelas et al. focused initial research studies using the CLIC framework on the racial identities of African Americans, although use of this framework could be extended to other racial identities, cultural identities, and gender identities. Academic identity signifies the positions and characteristics of a participant in academic classroom tasks. As much research exists on content learning, this study utilizes the identity types outlined in the CLIC framework to investigate the relationships between student identity construction and empowerment in the context of critical mathematics education.

Framing Empowerment

In conjunction with content learning and positive identity construction in a critical mathematics classroom context, students become empowered as confident, capable, and critical users and creators of mathematics. Ernest (2002) defines three domains of empowerment regarding mathematics education: mathematical empowerment, social empowerment, and epistemological empowerment. Mathematical empowerment implies establishing personal capabilities of the particular skills and practices of school mathematics. Social empowerment denotes both control over study and work opportunities and development as a critical mathematical citizen. Epistemological empowerment entails confidence in applying mathematics, as well as creating and validating mathematical knowledge. Ernest's domains of empowerment coincide with the purposes of high school mathematics: expanded academic and professional opportunities, deep and critical understanding of the world, and a confident and creative disposition towards mathematics.

Proposed Relationships Between Identity Construction and Empowerment

In the context of the critical mathematics classroom, evidence of student identity construction informs understanding of student empowerment. Specifically, the author suggests that the development of positive disciplinary, cultural, racial, gender, and academic identities leads to critical mathematics consciousness, thereby empowering students mathematically, socially, and epistemologically, to write the world.

Identity construction precedes empowered action. As Freire (1978/1972) states, “the form of action men adopt is to a large extent a function of how they perceive themselves in the world” (p. 72). In keeping, Skovsmose (2016) reminds researchers that empowerment can be interpreted in terms of both personal and socio-political identities. The following sections detail the proposed relationships between the development of different identities and student empowerment in different domains. These relationships are outlined in the Critical Identity Construction and Empowerment Framework (Crit-IC-E) shown Figure 1.

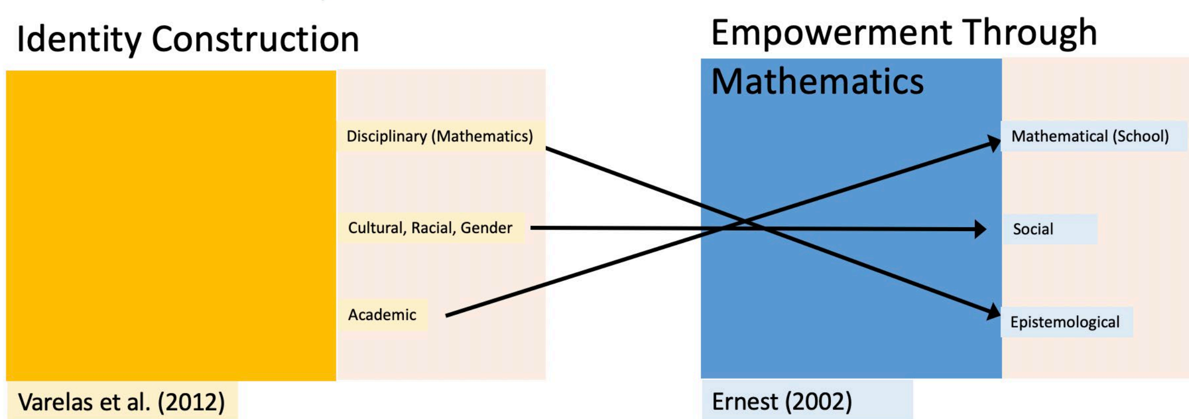

Figure 1. The Critical Identity Construction and Empowerment Framework (Crit-IC-E)

Numerous critical mathematics empirical studies highlight both student identity construction and empowerment, but do not necessarily emphasize the connection between the two concepts (Andersson, 2010; Dunleavy, 2015; Gutstein, 2016; Solomon, et al., 2011; Wright, 2016). However, a review of the literature suggests these connections exist and that attention to identity construction influences student empowerment. Results of analysis of the literature suggest that attention to disciplinary identity construction assists with social and epistemological empowerment (Braathe & Solomon, 2015; Gutierrez, 2013; Kokka, 2019; Mycyk, 2019). Similarly, attention to cultural, racial, and gender identity development permits mathematical, social, and epistemological empowerment (Kokka, 2017; Moreno & Rutledge, 2018; Sims, 2016). Studies emphasizing academic identity development correspond with mathematical and epistemological empowerment (Johnson, 2020; Raygoza, 2016; Tisch, 2014). Through further analysis in a critical mathematics context, this research hopes to clarify the relationships between the construction of different identities and the various domains of empowerment.

To follow, the author hopes to provide insight to critical mathematics educators in assisting students in developing critical consciousness. Murrell (1997) proposes the purpose of a liberatory education "is the cultivation of a consciousness and the development of children's identities, as well as academic proficiencies" (p. 28). This researcg suggests that the positive development of children's identities is equivalent to the development of critical consciousness. With this critical consciousness, students can be empowered to continually create and re-create knowledge and act on the world (Frankenstein, 1983).

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A Different Approach to Trigonometry

Sandra Fital-Akelbek, Weber State University

Since there is a shortage of students going into science and engineering careers, good knowledge of trigonometry increases students' ability to move into STEM professions. In a report to the president, the President's Council of Advisors on Science and Technology "*found that economic forecasts point to a need for producing, over the next decade, approximately 1 million more college graduates in STEM fields than expected under current assumptions".* The report also pointed out that "*fewer than 40% of students who enter college intending to major in a STEM field complete a STEM degree"* (PCAST 2012, p.7)

Trigonometry is an important subject that is required for serval science, biomedical, mathematics and technology courses. However, research has shown that trigonometry causes many problems and difficulties for students (Kendal & Stacey 1996, Weber 2005, 2008, Moore 2013).

Kendal and Stacey (1996) found that the unit circle approach gave students more opportunities to make mistakes. They showed that students who were taught the right triangle method performed significantly better than students who were taught the unit circle. However, Weber (2005) states that the unit circle was a more effective pedagogical tool than right triangles. He found that students were more likely to recognize sine and cosine as functions if taught using a unit circle approach. But he also admits that " *simply teaching about trigonometric operations by using a unit circle model does not guarantee that substantial learning will occur*". … "*the opportunity to think of sine and cosine as processes is critical*" … (Weber 2008, p.147)

The unit circle is a helpful approach to learning trigonometry and, if applied correctly, it can be a very good tool. Is it really necessary to memorize the unit circle the way students learn it at schools? Is it possible to teach trigonometry without spending time on making students memorize the unit circle? To find the answer, let us look at a different approach to trigonometry. We will share how we teach trigonometry in our classrooms. The approach is consistent with NCTM Principles to Actions (NCTM, 2014), where the learning is focused on meaningful tasks that promote reasoning and problem solving.

We start the trigonometry unit with basic trigonometric ratios and expand it to all six trigonometric ratios. At the same time, we emphasize the reciprocal functions: cosecant, secant, and cotangent as reciprocal of sine, cosine, and cotangent, respectively, that is

$$csc\theta = \frac{1}{sin\theta}, \qquad sec\theta = \frac{1}{cos\theta}, \qquad cot\theta = \frac{1}{tan\theta}.$$

Our focus in trigonometry is on the three primary trigonometric functions, sine, cosine and tangent. Knowing the values of sine, cosine and tangent allows us to find the values of all the other trigonometric functions. Therefore, our next task is to identify the values of trigonometric functions of the special angles: $30°, 45°, 60°$, and we recognize that: $30° = \frac{\pi}{6}\ rad,\ \ 45° = \frac{\pi}{4}\ rad,\ \ 60° = \frac{\pi}{3}\ rad$. This task can be done in several different ways. Using a creative approach, where students are asked to create triangles with specified features or simply solving special right triangles. Once the values are found, we summarize them in the chart below, Figure 1. The chart will play an important role in studying trigonometry, so we ask students to memorize it. We point out several features of the chart that can help students master it.

	$\frac{\pi}{6}$ 30°	$\frac{\pi}{4}$ 45°	$\frac{\pi}{3}$ 60°
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

numerators are: squre roots of 1, 2 and 3 knowing that $\sqrt{1} = 1$

all denominators are the same, 2

the row of cosine values is the same as sine, but written from the back

to find the tangent values, we divide the values of sine and cosine in each column

Figure 1. Exact values of the special angles for primary trigonometric functions.

In the chart, the angles go from the smallest value to the largest. The functions go in order: sine, cosine and tangent. In the first row of the table the values of sine all have denominators of 2, and the numerators are square roots of 1, 2 and 3, respectively, knowing that $\sqrt{1} = 1$. The second row of the table has values of cosine that are the same values as sine, but written backwards. The values of tangent can be obtained by dividing the values of sine and cosine, that is $tan\theta = \frac{sin\theta}{cos\theta}$. Since this trigonometric identity is taught later in a chapter, at this point, we just introduce the identity and we ask students to prove it (or justify it) for an acute angle only, using a right triangle and the trigonometric ratios.

Good knowledge of trigonometry requires students to know the graphs of the primary trigonometric functions. Thus, the values of the trigonometric functions of quadrantal angles, that is: $0°, 90°, 180°, 270°, 360°$ (or $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$) can be found from the graphs, Figure 2.

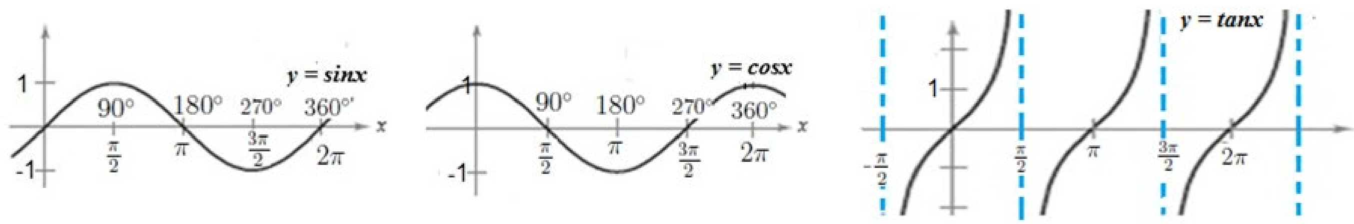

Figure 2. Graphs of the primary trigonometric functions.

From Figure 2, we can see that $sin 90° = 1$ or $cos\frac{3\pi}{2} = 0$ and $tan\frac{\pi}{2}$ does not exist.

Every angle can be placed in a coordinate system, Figure 3, using the origin as the angle’s vertex and the positive x-axis as the initial side of the angle. As we rotate two markers to show students the opening of the angle, we can identify the quadrant where the terminal side of the angle is located.

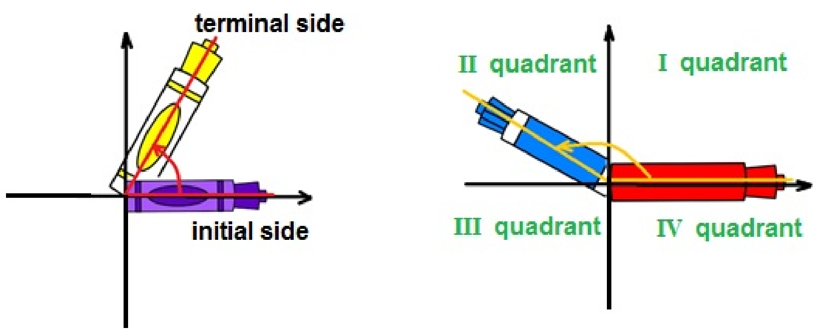

Figure 3. Angles placed in a coordinate system helps students identify quadrants as the angles are rotated.

The first angle in Figure 3 has its terminal side in the first quadrant, and the second angle has its terminal side in the second quadrant. (Quadrants are labeled counter clockwise starting from the top right corner, Figure 3).

We define a **reference angle** as an acute positive angle between the terminal side of the given angle and the x-axis, Figure 4.

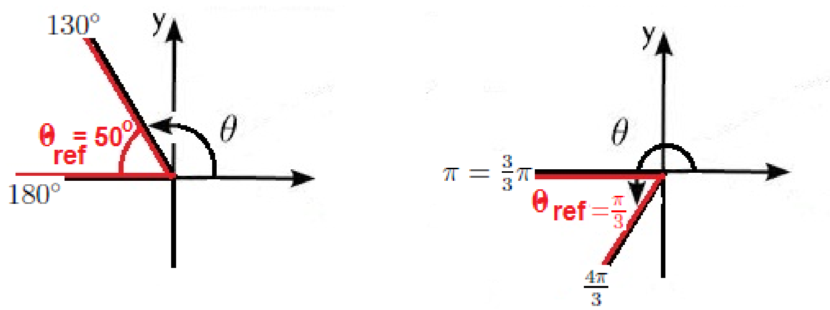

Figure 4. Reference angle

For example, the reference angle of $130°$ is $50°$, the reference angle of $\frac{4\pi}{3}$ rad is $\frac{\pi}{3}$ rad, see Figure 4, and the reference angle of $-\frac{\pi}{4}$ is just $\frac{\pi}{4}$. Using the radian measure is quite difficult for students, we try to simplify the process by using fractions. For example, as we rotate the angle of $\frac{4\pi}{3}$ rad (Figure 4), students noticed that the rotation passed the angle π. And since the given angle is a fraction of 4/3 of π, it is beneficial to change the whole π into $\frac{3}{3}\pi$. Then the reference angle can easily be identified as $\frac{\pi}{3}$ rad.

Our next task is to identify quadrants, where the graphs of the primary trigonometric functions are positive and where they are negative. It is beneficial to line up the graphs and identify the quadrants on the graphs as in Figure 5. Then we ask students to identify what graphs are positive in each quadrant.

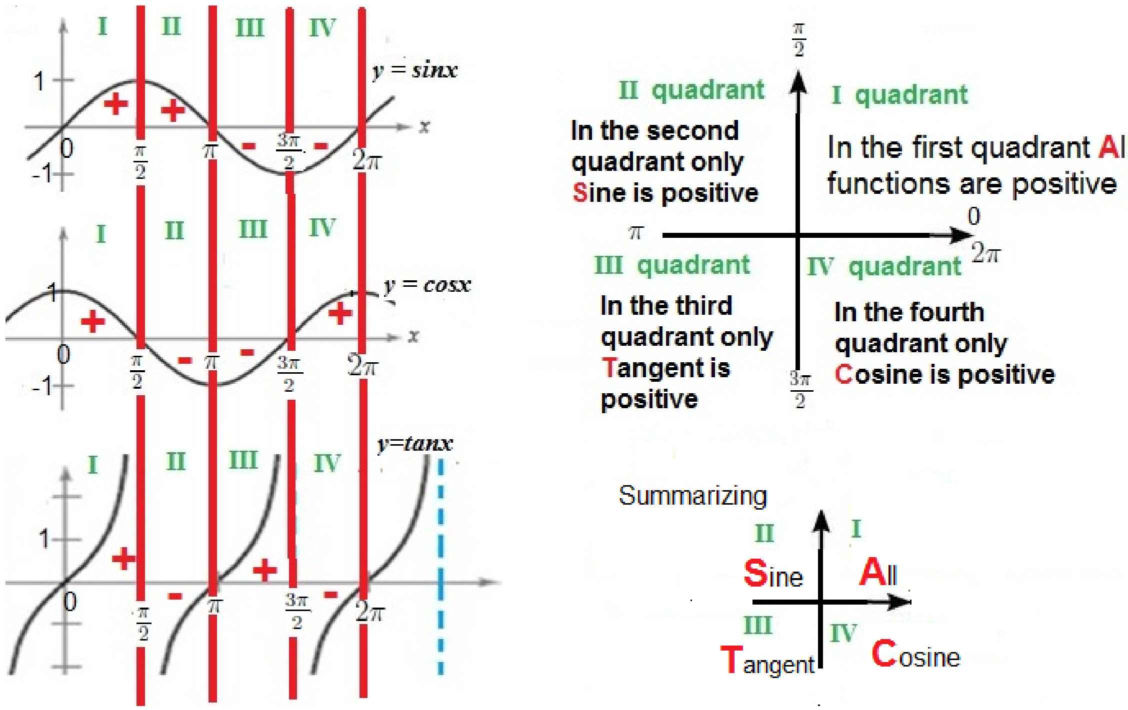

Figure 5. Positive and negative values of the primary trigonometric functions in each quadrant.

In the first quadrant all three primary trigonometric functions are positive, in the second only sine is positive (cosine and tangent are negative), in the third quadrant only tangent is positive and in the fourth only cosine is positive. We summarize it on the right side of Figure 5. We give students some time to think about the positive and negative values of the functions, and we ask them to find the easiest way in which they can memorize where the trigonometric functions are positive. After some time, students notice that it is sufficient to memorize the first letters: A, S, T, C and an acronym could help students to master it. For example, **A**ll **S**tudent **T**ake **C**alculus will summarize the letters A-S-T-C, or **A**dd **S**ugar **T**o **C**offee. Simple exercise, like identifying the quadrant where, cosine and tangent are negative, will help students utilize the chart in Figure 5.

At this point, students are ready to find the exact values of any angle. We use the following theorem:

Theorem 1. *All trigonometric functions of any angle θ are equal to the plus or minus of the same trigonometric functions of the corresponding reference angle. The sign, + or − is chosen according to the quadrant in which the given θ lies.*

In other words, to find the exact value of the given trigonometric function of a general angle, we first find the reference angle and identify the quadrant. Then by applying the above theorem we adjust the sign to + or − according to the quadrant, and find the exact value of the function using the chart in Figure 1. For example, to find the value of $\sin 210°$, students first find the reference angle to be $30°$ and identify the quadrant where the terminal side of the angle $210°$ is. The quadrant is the third quadrant, and by the above theorem we have, $\sin 210° = -\sin 30°$, using the chart in Figure 1 we can conclude

$\sin 210° = -\frac{1}{2}$. Multistep questions (like the one from Figure 6) can also be solved using the method discussed above. The question asks to find the exact value and simplify $4csc\frac{3\pi}{4} - \cot\left(-\frac{\pi}{4}\right) + cos270°$.

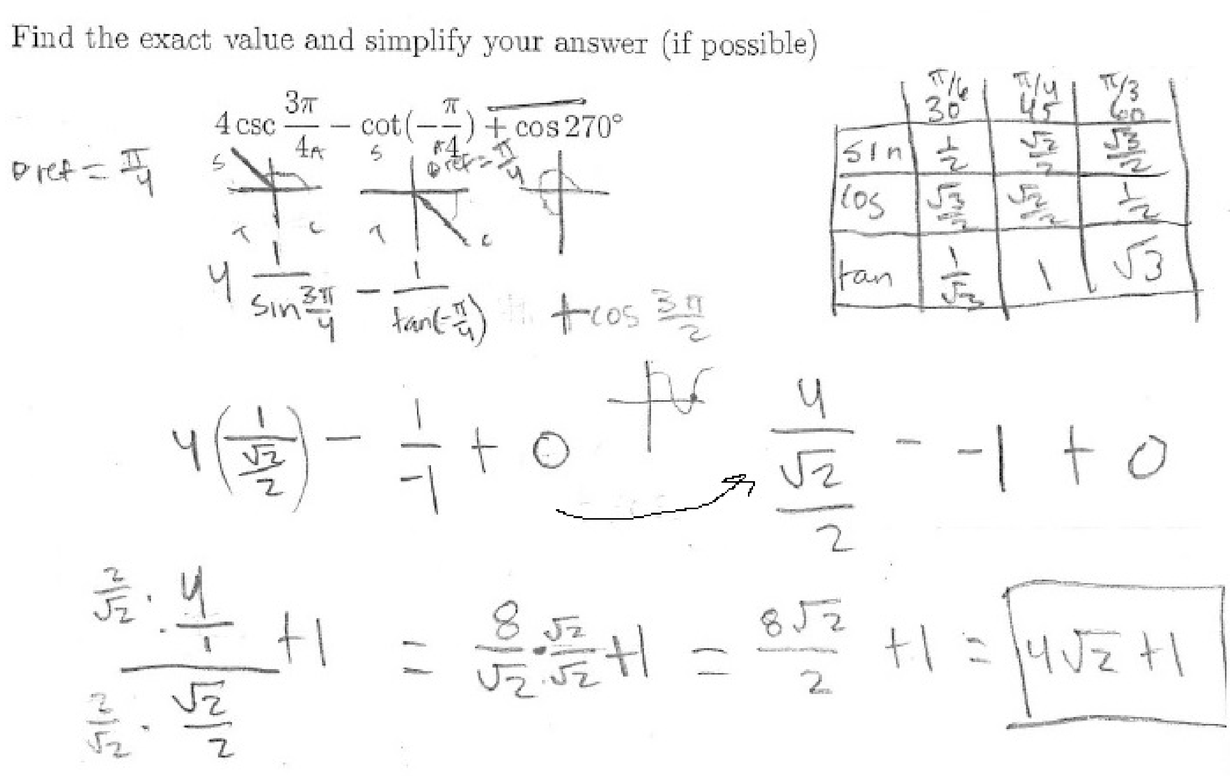

Figure 6. Student's work shows correct solution of the exam question using the method discussed above.

First, the student recognizes the reciprocal functions, that is $csc\theta = \frac{1}{sin\theta}$ and $cot\theta = \frac{1}{tan\theta}$. Next, the student placed each angle in the coordinate system to help them find the reference angle and identify the quadrant. For the angle $\frac{3\pi}{4}$ we can see that the reference angle is $\frac{\pi}{4}$ and the terminal side of the angle $\frac{3\pi}{4}$ rad is in the second quadrant (see Figure 6). Therefore, $csc\frac{3\pi}{4} = \frac{1}{sin\frac{3\pi}{4}} = \frac{1}{sin\frac{\pi}{4}} = \frac{1}{\frac{\sqrt{2}}{2}} = \sqrt{2}$. Thus, $4csc\frac{3\pi}{4} = 4\sqrt{2}$. For the second term, the student recognized that the angle $-\frac{\pi}{4}$ is in the fourth quadrant and the reference angle is $\frac{\pi}{4}$. Therefore, $\cot\left(-\frac{\pi}{4}\right) = \frac{1}{\tan\left(-\frac{\pi}{4}\right)} = \frac{1}{-tan\frac{\pi}{4}} = \frac{1}{-1} = -1$. The last term in Figure 6, is $cos270°$. The student noticed, that the angle $270°$ is a quadrantal angle, so the value can be found using the graph of cosine function (see Figure 2), and we have $cos270° = 0.$ Hence,

$$4csc\frac{3\pi}{4} - \cot\left(-\frac{\pi}{4}\right) + cos270° = 4\sqrt{2} - (-1) + 0 = 4\sqrt{2} + 1.$$

Once the students have good understanding of how the exact values of any angle can be found we give them time to think on justifying the above theorem. The first question we ask is: Why does sine of $135°$ have the same value as sine of $45°$, that $sin135° = sin45°$, or using radians, why $sin\frac{3\pi}{4} = sin\frac{\pi}{4}$? The answer that we quite often hear is: "*Because the reference angle of* $135°$ *is* $45°$, *and the angle is in second quadrant, so sine is positive.*" The teacher would say: "*That is very true, because we applied the theorem, but we are trying to figure out, why the theorem is true? Can we find a different way to justify the statement: why the sine of an angle in the second quadrant is the same as the sine of the reference angle?*" In a leading discussion the teacher might need to suggest using trigonometric graphs. After some time students notice that the reference angle is a positive angle in the first quadrant. We look at the piece of the graph of sine that is in the first quadrant. We use the part of the graph from the first quadrant to justify the values of sine of any angle as in Figure 7 below.

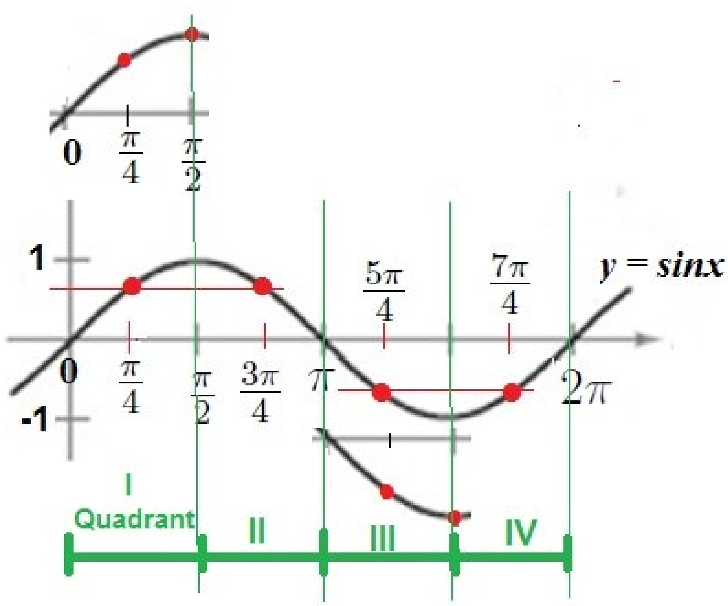

Figure 7. The value of sine of any angle is plus or minus the value of sine of the reference angle.

Another example of student's work is shown in Figure 8. The question asks to prove the identity $cos(180° + \theta) = -cos\theta$. The student applied one of the sum and difference formulae, namely, $cos(\alpha + \beta) = cos\alpha \cdot cos\beta - sin\alpha \cdot sin\beta$. Using the graphs of the sine and cosine functions we can find the values of $cos180° = -1$, and $sin180° = 0$, and the identity follows immediately (see Figure 8).

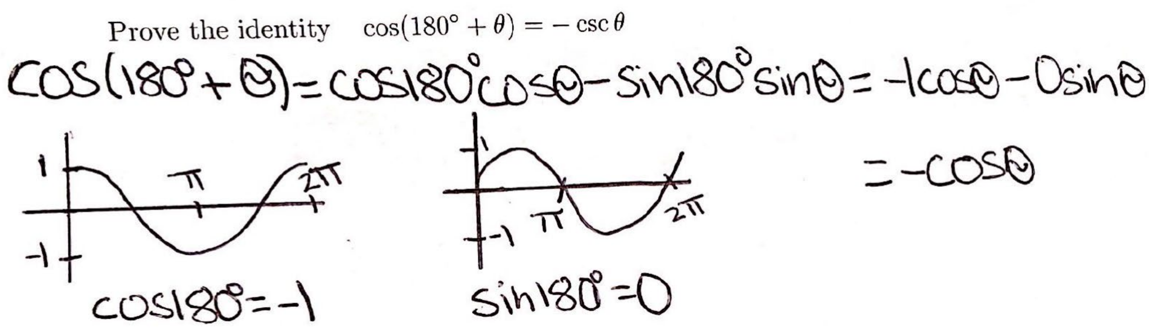

Figure 8. Student's work of proving trigonometric identity.

Solving trigonometric equations is a major skill for the trigonometry unit. We begin solving trigonometric equations with the simple example as in Figure 9.

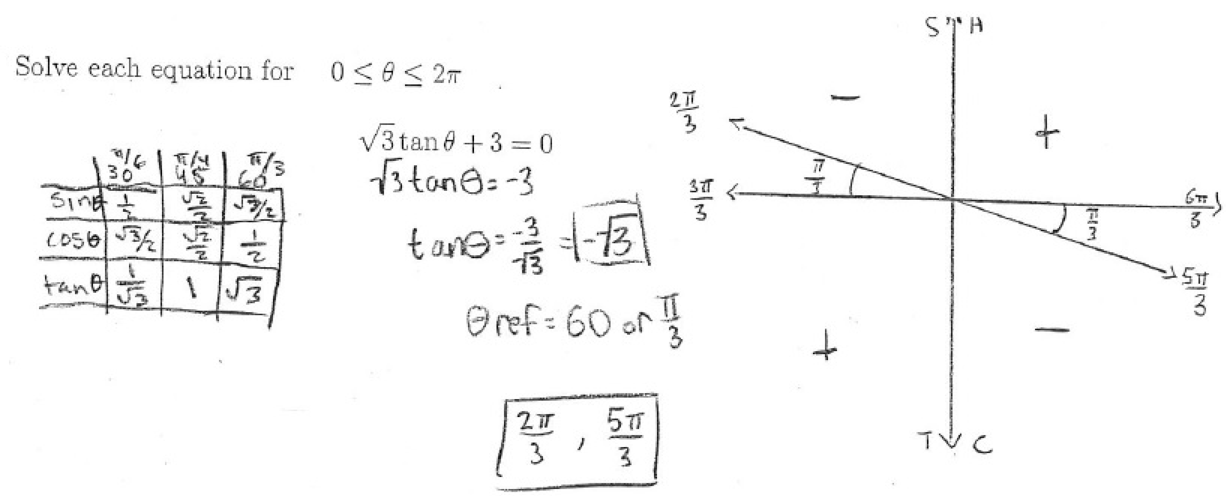

Figure 9. Student's solution of a basic trigonometric equation.

The question asks to solve the trigonometric equation $\sqrt{3}tan\theta + 3 = 0$ for $0 \leq \theta \leq 2\pi$. First, we isolate the trigonometric function to $tan\theta = -\sqrt{3}$. Using Theorem 1, we know that tangent of any angle is equal to the plus or minus tangent of the reference angle. The chart in Figure 1 helps us to find the reference angle, that is, we look for an angle where $tan\theta = \sqrt{3}$ and we establish the reference angle to be $60°$ or $\frac{\pi}{3}$. Next, we can identify the quadrant where the tangent is negative using the chart in Figure 5. Tangent is negative in quadrant II and IV (also see student's solution in Figure 9). Therefore, solutions to the equation are two angles: one is an angle with the terminal side in the second quadrant whose reference angle is $\frac{\pi}{3}$, and the second is an angle with the terminal side in the fourth quadrant whose reference angle is $\frac{\pi}{3}$. Changing π and 2π into $\frac{3}{3}\pi$ and $\frac{6}{3}\pi$, respectively, the student (Figure 9) finds the correct answer $\frac{2\pi}{3}$, $\frac{5\pi}{3}$. Once the equation is solved, we ask students to look at the graph of the tangent function, because the equation $tan\theta = -\sqrt{3}$ can be solved graphically as in Figure 10.

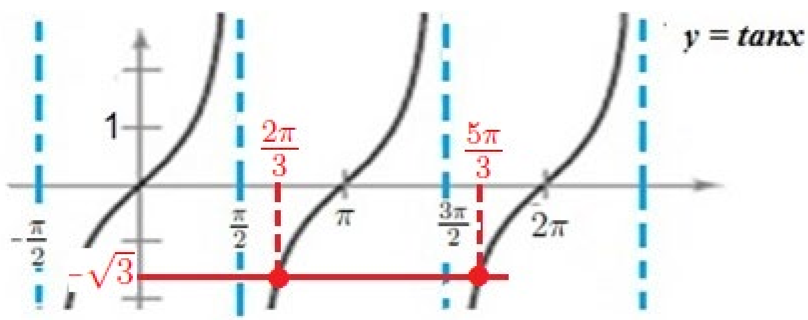

Figure 10. The graphical solution to the equation $tan\theta = -\sqrt{3}$.

Another example of a trigonometric equation is shown in Figure 11.

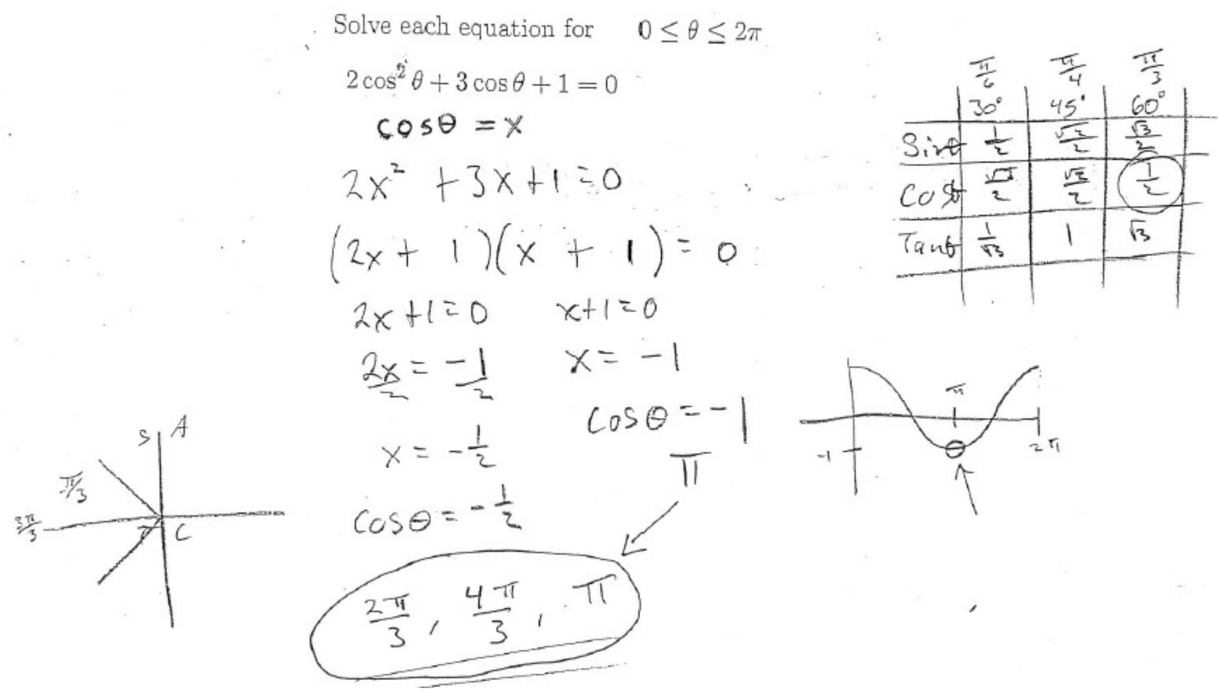

Figure 11. Student's solution to a multistep trigonometric equation.

Solve the given equation $2cos^2\theta + 3cos\theta + 1 = 0$ for $0 \leq \theta \leq 2\pi$. Looking at the solution in Figure 11, the student first used substitution $cos\theta = x$ to solve the quadratic equation: $2x^2 + 3x + 1 = 0$. The solution is $x = -\frac{1}{2}$, x = -1. Therefore the original equation can be simplified to two trigonometric equations: $cos\theta = -\frac{1}{2}$, $cos\theta = -1$. The second equation, $cos\theta = -1$, is solved using the graph of the cosine function. The solution is π. The first equation, $cos\theta = -\frac{1}{2}$, is solved as the example in Figure 9. The student found the reference angle using the chart in Figure 1, to be $\frac{\pi}{3}$, and identified the quadrant using Figure 5, that is, cosine is negative in the second and third quadrant. Using Theorem 1, the student found the angles with terminal sides in the second and third quadrants, respectively, whose reference angle is π/3. To help identify the angles, the student changed π into $\frac{3}{3}\pi$, and the solution is $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, see

Figure 11. Thus, the solution to the original equation $2cos^2\theta + 3cos\theta + 1 = 0$ for $0 \leq \theta \leq 2\pi$, is $\frac{2\pi}{3}, \frac{4\pi}{3}, \pi$.

The above discussion shows that trigonometry can be taught without making students memorize the unit circle. This different approach to trigonometry, just shifts the focus from the unit circle to the graphs of basic trigonometric functions and uses properties of trigonometric functions that several teachers are already using at schools.

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A Math RTI Structure to Support Equity and Learning for all Students

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Abstract

This piece presents an effective mathematics Response to Intervention (RTI) structure implemented by a collaborative educational team at a university laboratory school. The team consists of teachers, a university member, student teachers, support staff, and administration. The team takes on the challenge of restructuring their RTI model to effectively help all students feel valued in their mathematics learning and to show mathematics learning growth. The general classroom teachers had three goals. They were interested in developing stronger communication between the large cohort of educators, increasing the use of data to drive decisions, and finding ways to help students feel valued and purposeful in their mathematical knowledge. Considering these goals, the collaborative team decided to (1) regularly hold professional learning community (PLC) sessions, (2) use common forms of assessment (CFA), formative teacher data, and proficiency scales, and (3) implement mathematics journaling and common robust math tasks (CRMT) to unify students. This piece provides details of the new math RTI structure as well as an example of the effectiveness of the structure. There is an overall excitement from all stakeholders about the new structure, and most of all, students are enjoying RTI math time, are feeling more valued in their groups, and are demonstrating positive learning results.

A Math RTI Structure to Support Equity and Learning for all Students

I was recently inspired by an email I received from a 6th-grade teacher regarding her desire to meet and discuss ways to restructure her 6th-grade math Response to Intervention (RTI) instruction. This portion of instruction is in addition to the whole group instruction, and meant to (a) intervene for students who have not yet grasped the content, and (b) enrich the knowledge of students who already demonstrated proficiency. Already, the 6th-grade math cohort had demonstrated strong math learning trends on previous measures, however this teacher was excited and eager to find ways to further improve instruction to benefit students.

I met with Mrs. Smith (pseudonym) on a Friday and we brainstormed ways to restructure math RTI time to meet students' needs; we blended my knowledge in the research world and hers in the practitioner world. After the hour-long brainstorming session, we identified three specific goals that Mrs. Smith (representing the 6th-grade math RTI team) wanted to achieve. These three goals were (1) increase communication and cohesiveness between all the different group instructors, (b) bolster the use of data to drive instructional decisions, and (c) increase the students' disposition about their value and purpose in mathematics.

Goal #1: Increase Communication and Cohesiveness of RTI Math Instruction Across Groups

The first goal of the 6th-grade team was to improve math RTI instruction by developing cohesiveness between topics from general math instruction and RTI math instruction, and increasing communication between the various math RTI small group instructors (e.g., student teachers, specialists, para-professionals). The connection between the general math instruction and the math RTI instruction was minimal, and decisions made in one or the other seemed independent of one another. As a result, math RTI only partly supplemented learning taking place in the general math class instruction. Additionally in the current structure, the entire 6th-grade student population was divided into nine small groups, each with an instructor. Each math RTI small group instructor would create their own lessons and address the skills they deemed important. However, the 6th-grade team wanted to improve communication between all these different groups so the large team functioned like a single unit. Increased communication across group instructors would not only support cohesiveness, it would help some of the instructors in teaching roles who may not have had the pedagogical content knowledge (Ball, Thames, & Phelps, 2008; Shulman, 1986) to implement high quality 6th-grade instruction. Primarily, it seemed ineffective to put a student teacher with a group of 6th-graders with minimal direction or scaffolding of instruction.

Goal #2: Bolster the Use of Data to Drive Decisions on Groupings and Instruction

The second goal was to increase the use of data to drive decisions on how to place students in their math RTI small groups and what should be taught in each group. In the past, each math RTI small group's instructor would decide what their students needed, or just teach a general lesson that matched the school's scope and sequence. However, each student would remain in the same math RTI small group. This meant that regardless of whether or not a specific student excelled with a skill or didn't quite understand a concept, there was little fluidity between groups, meaning the student may stay in a math RTI small group that was not meeting their specific needs most effectively. In a perfect world, each math RTI small group would be compiled of students who demonstrated similar strengths and weakness within a certain topic as substantiated by a common form of assessment. The 6th-grade team decided that a formative and summative measure to substantiate student placement in each math RTI small group would be critical. This data would prove useful in trying to meet the students where they were at and help all the students succeed.

Goal #3: Increase Students Sense of Value and Purpose in Mathematics

The final goal of the 6th-grade team was to find a structural way to help all students feel like they had a purpose and sense of value in mathematics. There was a growing realization that there may be an equity issue in the current structure. The students in the lowest groups were consistently in these groups and they were feeling as if they were 'bad' at math. There was little to no mobility between the groups. There had been growing concern among some parents that their children were being pulled away from everyone else and were being given 'remedial' work. It was important to this 6th-grade team to find a way to ameliorate this issue and provide experiences for all students to work with various teachers and with a variety of peers.

Instructional Implementations to Meet these Goals

With these three major goals in mind, we were dedicated to developing a new structure that could offer possible solutions and help all students regardless of level of proficiency with a certain mathematical skill, achieve growth and a sense of value. In the end, we decided on a plan (Figure 1) that may achieve the proposed goals. To meet goal #1, we organized weekly professional learning community (PLC) sessions to increase communication and cohesiveness among the RTI instructional team. To meet goal #2, common assessments, proficiency scales, and the use of formative teacher data were implemented to drive instructional decisions. Finally, to meet goal # 3, mathematics journals and common robust math tasks were implemented to unify groups and ensure all students were exposed to rigorous and meaningful mathematics.

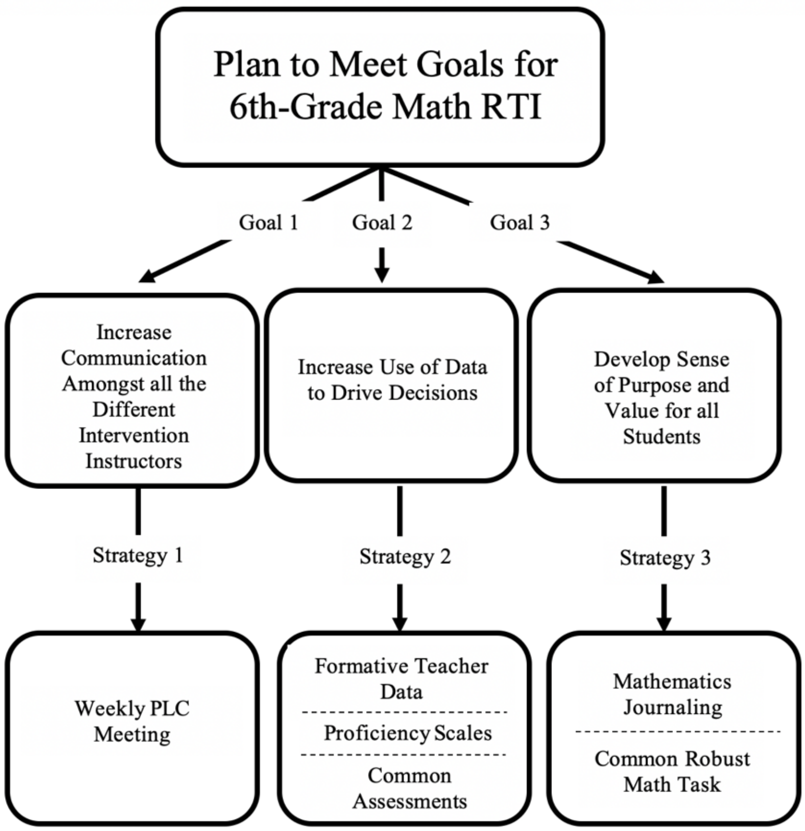

Figure 1: Plan to Meet Goals for 6th-Grade Math RTI

Strategy #1: Professional Learning Community Sessions

One of the elements to the newly proposed structure was the weekly professional learning community session which involved the entire team of educators who taught 6th-grade math RTI small groups (Baker et al., 2018). This meeting was scheduled for every Friday at the conclusion of the week and had two primary goals; (a) to share formative data and adjust math RTI small groups accordingly, and (b) to launch the new common robust math task (CRMT) that would be implemented in each group in the following week.

Adjusting Groups Based on Formative Data

Maintaining fluid groups was a key to both ensuring equitable access to mathematics and to ensure students were having the opportunity to work with various teachers and collaborate with various students. We feared, as it has been the case in many differentiated programs, the groups would become stagnant and the members would remain in the same groups for months, semesters, or even years, regardless of the progress they were making toward learning outcomes. This is why formative data was collected and evaluated weekly in form of anecdotal teacher notes, student artifacts, and student math journals. The formative data obtained from these sources helped ensure that students were immediately shifted into appropriate groups to receive the specific instruction they needed.

Professional Collaboration on a Common Robust Math Task

One of the most exciting parts of this PLC structure was to launch the new CRMT that would be implemented across groups for the following week. This essential collaboration served as a type of a think tank session where all of the teachers would begin to brainstorm ways to enrich or adapt a task that was common across groups in order to meet the different needs of the students. One of the most critical outcomes of this was that the staff who lacked certain pedagogical content knowledge were engaged and enlightened by all the dialogue and ideas that would flow from various professionals.

Strategy #2: Formative Teacher Data, Proficiency Scales, and Common Assessments

The strategies implemented to try and increase data driven decisions was the implementation of common formative and summative assessments as well as proficiency scales. New assessment specialists emphasize the importance of teacher formative data (Marzano, 2011). This being the case, an emphasis was placed on teachers having documentation to inform their formative evaluations. Each math RTI small group instructor would bring their corresponding students' math journals, along with teacher anecdotal notes, to the weekly PLC. These served as the documentation of formative assessment and helped substantiate any movement of students. Additionally, a common form of assessment (CFA) was administered before the RTI groups began to address a concept (pre-assessment) and then administered at the end of the session on that same concept (post-assessment). Ultimately, the pre- and post-summative common forms of assessments were important to gain a "big-picture" of classroom development, but the weekly teacher formative assessments were critical to make student specific and immediate modifications.

Proficiency scales also proved important to help break mathematics topics down into their basic, foundation, proficient, and advanced concepts. These scales helped the teachers in each group know exactly what skills within the concept the students in that group were struggling with and needed to work on. It also allowed the teachers to see the skills at the next level of proficiency so they could help develop knowledge with those skills.

Strategy #3: Common Robust Mathematics Tasks and Mathematics Journaling and

The final elements to the new structure was the implementation of CRMTs and mathematics journaling. In order to help all students feel like they had access to rich mathematics and to feel connected and equitable, we decided that offering all students the same CRMT in a similar format was essential. The key to these CRMTs was to offer robust tasks that had high ceilings and low floors, in other words, multiple access points (Boaler, 2016). These multiple access points meant that there were ways in which the task could be enriched for certain students and ways they could be used to reinforce fundamental concepts for other students. These CRMTs were all based on real-life, required much more than right or wrong responses, and required students to consider context while problem solving. This idea of having all the students work on the same CRMT was elemental to the new structure. It allowed all students to feel like they were working on the same important mathematics topics. Little did each group know that the groups around them may have attacked aspects of the task that they may not have. This ensured that within groups, students were being instructed within their Zone of Proximal Development (Vygotsky, 1967/1922) while between-groups, they still felt a sense of connectedness and value. The second section of this article provides an example of a CRMT created for the students that relates their recent experiential learning rafting trip to mathematics.

The students also maintained a math journal in their RTI small groups that corresponded with each weeks CRMT. All the students would begin the week by taping the exact same image in their journal and labeling the page with the title, skills, and date. As the week progressed, each student's journal across math RTI small groups would begin to look different based on the specific skills of that group. These journals were important to the formative assessment process (they documented evidence of proficiency), important as a creative artifact of student mathematical thinking, and served as an important math tool (students would often refer back to their journals in subsequent weeks to help them remember an idea or strategy). Finally, the math journals that corresponded to the weekly CRMT allowed for the students to feel united to one another. Regardless of which math RTI small group they were in, they had similar skills, images, and topics in their journals. This allowed for a certain connectedness between groups and made the students feel proud of the mathematical work they had achieved.

A New Structure for Success

In order to implement all these program changes, an innovative and creative classroom structure was needed. We continued to discuss logistics such as resources, space, transition times, and practicality, until we decided on a structure that would help us embed the new program implementations. Figure 2 illustrates the final 6th-grade math RTI.

The learning outcome for two consecutive weeks of math RTI instruction was always

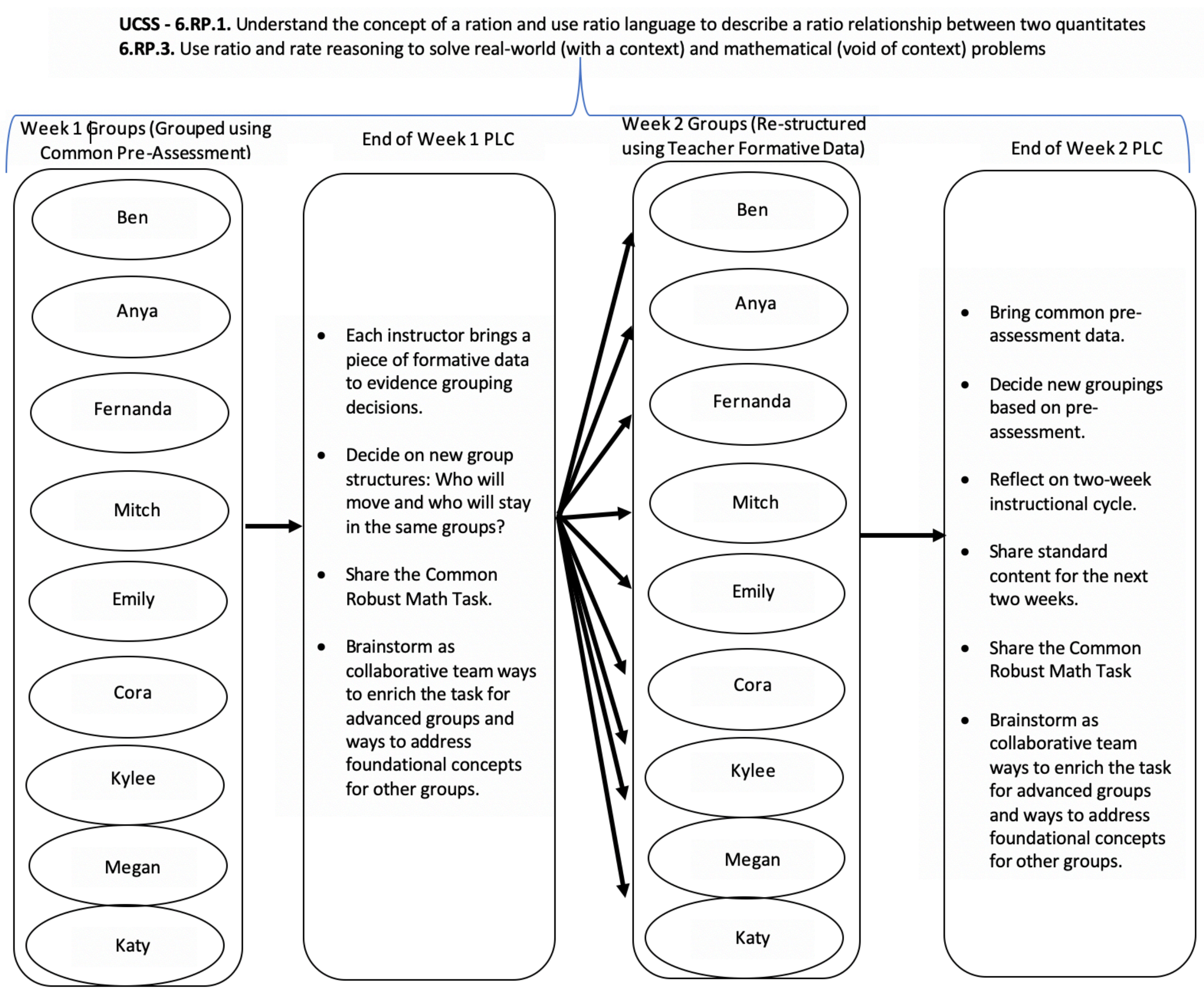

Figure 2: Final Structure for Math RTI

determined by a post-assessment that occurred during the general classroom math session. This helped ensure that the initial groups were appropriately organized, using the common post-assessment that was given after the general classroom math instruction but before the RTI instructional period. The nine groups all engaged in the same CRMT but each group would access the task's mathematics in very different ways. At the end of the first week, a PLC would occur to use formative data to shift students and then the new task for the following week (focused on the same learning outcome) would be launched and thoroughly discussed as to how it could be enriched or modified to address fundamental mathematics concepts. The next week would involve the new groups engaging in the new CRMT. At the conclusion of the week, the teachers would give the common post-assessment on the learning outcome. These data would be discussed at the PLC and decisions would be made on how to meet the needs of the remainder of students who had still not met proficiency on that topic. If it was many students, more weeks could be dedicated to the learning outcome. If it was just a small number of students, discussions and plans would take place accordingly as how to meet that specific students' needs.

Where the Rubber Meets the Road

With a great implementation plan in mind, the team prepared for Monday and the launch of the new program. The team had a PLC on the Friday before to finalize ideas on the CRMT, to unify ideas on the math journals, and to finalize initial groupings. Although there was some natural anxiety about trying something new, there was an overarching positivity about the new structure and everyone seemed excited to see what would happen.

My Role: Getting to put the Lesson into Practice

One of the group instructors was not at school for the first week, which gave me the opportunity to be the in charge of that group myself. This was an exciting opportunity because it meant I would get to see and feel first-hand how the new math RTI program ran rather than just hearing or seeing from afar.

My group consisted of seven students who were considered right at the proficient level for ratio and proportional understanding – the learning outcome for the two-week progression. The CRMT for the week was designed to bring in the students' knowledge about a certain river they had just finished rafting on a schools experiential learning trip. Indeed, they didn't just raft the river, they camped and rafted for the entirety of the week! Either way, they were intrigued that their CRMT was to look at the real-time data of the river flows to conceptualize rate, ratios, and proportions.

Day One: Making Sense of a Cubic Foot of Water and Rate of Change

The first of the four days consisted of the students taping in an anchor image (a map of the river) into their journals. They continued by discussing what could change the flow of the river (e.g., run-off, springs, irrigation, more/less downstream) and examined graphs on Chromebooks that showed the real-time flow of different locations along the river. Flow is typically measured in cubic feet per second (cfs) which was a foreign unit of measurement to the students. This required the students to debate what this really was and meant. They first had to justify to each other that a cubic foot was different than both a foot and a square foot, this was a 3-D unit of measurement. Next, they arrived at the consensus that a cfs was a ratio representing rate, 1 cubic foot:1 second. After this, the students began to record the different rates of flow at the different river flow recording stations. At this point all the measurements were in the unit rate form of 1 second: x cubic feet. This concluded our short 30-minute session.

Day Two: Finding Unit Rates and Comparing

Day two consisted of students using the unit rates of the different locations to evaluate how much water would be flowing down the stream after certain amounts of time, taking the amount of water that has flowed in a certain amount of time and finding the unit rate, and comparing rates and amounts of water at different locations. During day two, students completed valuable calculations that demonstrated proficiency at taking a ratio such as 5,284 cf:3s, and converting it to a unit rate, 1761.33 cf:1s. Then they would compare multiple sites that were all converted into unit ratios. This knowledge of how to find and apply unit rates to compare ratios would be considered the proficient level for 6th-grade (CCSSI, 2010).

After day two, it was time to enrich the students by embedding this ratio knowledge in a contextual problem that required creating their own ratios and by translating between different ratios.

Day Three: Launching a Problem-Solving Story

Figure 3 represents the story problem that was posed to the students at the onset of day three. As can be seen in the problem, it is contextually based in the CRMT. The problem also adds a layer of complexity that pushes the students to both use the online graph of water flows and begin translating their ratios between three different variables, time, cubic feet, and height. The students were excited to begin the problem and even drew the reservoir on the map in their journal to get an idea of what it would look like and how it would connect to the river. They organized into pairs and groups and vehemently began to attack the problem. One of the most encouraging moments during this phase was the variety of ways the students began to tackle the problem. Each group had different strategies. I even stopped the group at about 15 minutes and highlighted the approaches that each group was using. You could see that they found value in the other methods and one of the groups who had been trying to implement a slow (yet accurate) method, ended up changing strategy because they said, "Wow! The way they were doing it is

> A rancher digs a huge pit with the goal of turning it into a reservoir. He digs a diversion from the pit to the edge of the river and puts up a huge gate. When he opens the gate, the entire river is channeled into the diversion. He wants to fill his reservoir up 8.5 ft. but he doesn't want to over or underfill it. He knows that _____ cf of water fills his reservoir up 3 inches. So, he cautiously decides to open the gate for exactly 8 seconds as a start. Figure out if he has overfilled his reservoir or if he needs to open it longer in order to get the level of the reservoir to exactly 8.5 ft.

Figure 3. The story problem for day three and four.

going to be way faster than ours!" This type of mathematical reasoning and justifying is the exact type of thinking that National Council of Techers of Mathematics calls for in the math practices (NCTM, 2000). These CRMTs with a journaling element proved important in eliciting many of the math practices that are valued so highly in mathematics education. The students continued to work on the problem and as the session ended, they were wondering if they could finish. I told them we would come back to finish it on the next day.

Day Four: Completing the Problem-Solving Task and Sharing Results

The students jumped right back into their groups and finished work on the story problems. One of the biggest conceptual struggles the students had was making the transition between rate of water coming into the reservoir and comparing that to the change in height of water in the reservoir. All three groups didn't recognize the need to convert the water rise in the pond to a unit rate for water entering the pond. Although the groups effectively kept the 3 in:___ cf ratio, it caused more laborious work and somewhat less efficient strategies to conclude a final height of the water in the reservoir. Regardless, the student groups arrived at solutions and each were able to give short descriptions of both their answers and solution strategies. In this case, there were groups that had conflicting answers which elicited some fascinating justifications from both groups. After they had discussed their approaches, one of the groups figured out they had mistakenly used three inches of height increase as the unit rate, resulting in a figure that was three times the amount of the actual solution. This concluded one week of instruction in the new RTI structure.

End of the Week PLC meeting

There was so much positivity at the end of the first weeks PLC meeting. Teachers brought suggestions for student movement (formative data), enthusiasm about how the week went with their students, and creative ideas when the new CRMT was launched for the next week (creative task collaboration). One of the 6th-grade teachers commented on how, by having the common task, she could refer back to the concepts (cubic feet) in class and all the students knew what she was talking about and felt like they held valuable knowledge. Additionally, the kids were excited. During the hustle and bustle of students transitioning into and out of the RTI time, there were discussions taking place and excited comments from students about what they did in math RTI. It was clear that across groups, they were talking about the CRMT that they had worked on!

A Structure Moving into the Future

Although new, this structure has been powerful to the 6th-grade math RTI instruction. Not only did it provide a consistent structure which allowed the entire 6th-grade math team to communicate and teach consistently, it provided a climate where all students felt as though they had access to valuable mathematics and they were getting the precise instruction they needed. The structure is now undergoing it's second year of implementation and it is gaining recognition from both administration in the building and district. The purposeful and consistent use of data to drive decisions, PLCs to unify instructional teams, and common tasks to promote unity, is even prompting schoolwide discussions on how this structure may be implemented in language arts and other RTI instructional times to support the same types of outcomes.

Basic Components for Implementing this Structure

For anyone interested in incorporating this structure, we warn that not each piece will be perfectly implemented immediately. Figure 4 highlights the three components of this structure along with their level of implementability. We recommend starting off with the immediate tangible elements that can be incorporated (i.e., math journaling, CRMT). As teachers are more and more comfortable with the implementation, they can begin to transition into some of the more intermediate structural and long-term assessment-based changes.

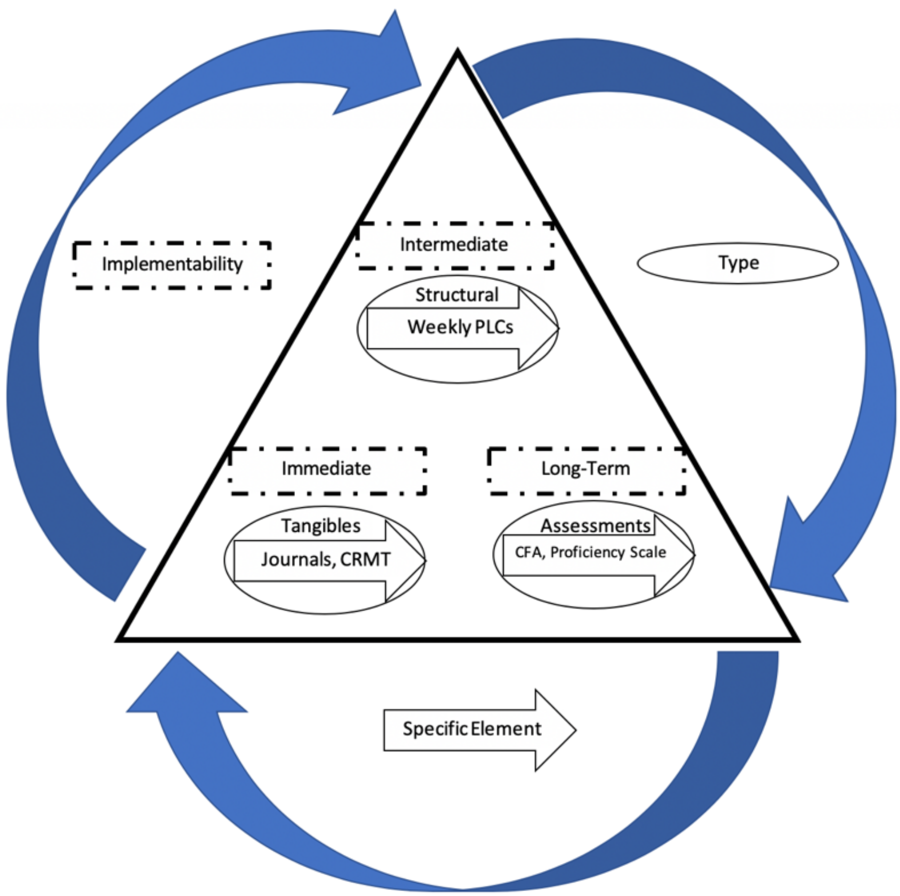

Figure 4. Teacher guide to practical implementation of new math RTI structure.

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The Lost Voices in Mathematics Teaching

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There are many factors that influence mathematics teaching. Technological advances, class sizes, budget cuts, standards, and high-stakes tests are just some of the many factors that could inspire change in how we teach mathematics. However, in most of the teaching reforms of the past, students' voices seem to be lost in the shadows. We must wonder; how do students want to be taught? It is certainly possible for an adult learner to be aware of conditions or instructional methods that facilitate optimal learning for him or herself. These preferences may all be different and hard to synthesize into one instructional method, but is that enough reason to abandon their preferences altogether? Even in a dichotomized case where a student is to choose between Direct Instruction (DI) and Discovery Learning (DL) there can be interesting answers.

One of my students said, "I don't like being the teacher. I don't like figuring it out on my own and then having to explain it to others. I pay tons of money to learn and not to teach – it's not my role as a student." On the other end of this dichotomy, another student expressed, "it's always nice to come up with the math rules, it makes me understand and remember even more." This exemplifies students' opinions on instruction, but the reality is that this is not a simple dichotomy of DI versus DL, rather, it can be viewed as a continuum ranging from DI to DL. Nevertheless, in order to explore the students' voices, I introduced DL and then asked them to compare it to DI (my usual teaching method) in a dichotomized approach.

Background

Mathematics has traditionally been taught through DI. DI is a teaching method "… that entails clearly defined learning outcomes, proceeding in small steps, active student participation, systematic feedback, guided and independent practice, and checking for student mastery" (Cooper, Heron, & Heward, 2014, p.6). The advocates of DI state that the method is efficient and it allows greater coverage of topics, in a short period time, than other instructional methods. However, many critics of DI claim that mathematics curricula based on DI encourages learners to take a passive role rather than an active one. Furthermore, the critics argue that this traditional way of teaching highly focuses on drills and memorization, and it fails to foster conceptual understanding. Among these critics is Alan Schoenfeld who prefers to teach mathematics in an exploratory way through problem solving (Schoenfeld, 1992). Many modern reformists endorse teaching methods such as DL, which typically includes problem-based instruction, inquiry, experiential, or constructivist learning (Kirschner, Sweller, & Clark, 2006).

Discovery learning is when the learners acquire knowledge on his/her own facilitated by teachers' tasks and questions. While the extent of guidance in DL might be less than that of DL, the teacher must still provide carefully guided tasks, examples, questioning strategies, and feedback if DL is to be effective (Alfieri, Brooks, Aldrich, & Tenenbaum, 2011). Schunk (2016) describes it as constructing and testing hypotheses rather than passively reading or listening to teacher presentations. The student is guided through inductive reasoning until they are able to formulate rules and principles on their own (Kirschner, Sweller, & Clark, 2006). The rise of this instructional method can be traced to the post-Sputnik science curriculum reforms where educators were switching to the notion that knowledge is best obtained through experience as governed by the subject. But yet again, even this instructional method has its critics. Kirschner, Sweller, and Clark (2006) cited several studies (for example, Gollub, Berthanthal, Labov, & Curtis, 2003; McCray, DeHaan, & Schuck, 2003), with empirical evidence, that state minimal guidance instruction, a fundamental characteristic of discovery learning, does not work. All these studies suggest that not only is minimally guided instruction less effective; it can also be detrimental since students can create misconceptions or incomplete or disordered knowledge.

Exposing my Students to Discovery Learning

With these two instructional approaches, I wanted to know what the learner would contribute to the conversation. The purpose of this paper was to explore students' voices as they compare two instructional methods (DI versus DL) when learning rules of exponents. These are developmental mathematics students that are enrolled in an intermediate algebra course with the youngest student being eighteen years old.

Our Typical Instructional Format

It is important to know that I usually teach these students using DI. Similarly, to Bender's (2009) guidelines, my DI usually follows this outline: 1) I begin by gaining students' attention, then I introduce the lesson by first relating it to previous topics. In this initial step, I use a lot of questions to activate students thinking as I orient them to the topic. 2) In the next step, I demonstrate the topic by using several examples and emphasizing that students should model the procedures. I also bring students' attention to any part that they may find difficult. 3) I then issue practice problems and supervise the students as they do them. I usually walk around the class to see how each student is faring with the practice problems and assist those that are struggling. In this phase, students are certainly allowed to consult a classmate if they need help. 4) I then issue more practice problems in the form of homework but this time the students are required to do it independently. 5) Lastly, I assess students' performance on independent work and reteach the skills to the students that are still struggling. This describes my typical teaching approach and that of my colleagues.

Planning a Discovery Learning-Based Lesson

In an attempt to introduce my students to DL, I used Calleja's (2016) guidelines for DL/Inquiry based learning. He explains a typical inquiry/discovery lesson involves task presentation – where the teacher offers a problem with clear expectations, but leaves the challenge to students to solve it. Then students collaborate in small groups to unravel the problem and, finally, they present their work to the whole class. Using these guideline, I designed worksheets to help the students discover the rules of exponents. Figure 1and 2 show two sample worksheets that were designed to provide students opportunities to discover the rules of exponents.

In designing the DL lesson plan, I employed Smith and Stein's (2011) levels of cognitive demand. Particularly, the worksheets were meant to elicit higher levels of cognitive demand by prescribing a pathway that would deliver them to the conceptual idea of creating the rules of exponents. The worksheets allowed them to explore the computations until they recognized a pattern and then expressed that pattern with symbols rather than numbers. The jump from concrete computations to an abstract representation of the rules required considerable cognitive effort. Once they proposed a rule, the students had to analyze its validity by attempting to find counterexamples until they were satisfied.

The expression	Write the expression in expanded form	The simplified expression
$\frac{2^5}{2^3}$	$\frac{2 * 2 \cancel{* 2 * 2 * 2}}{\cancel{2 * 2 * 2}}$	2^2
$\frac{3^8}{3^4}$		
$\frac{4^7}{4^5}$		
$\frac{9^7}{4^5}$		
$\frac{a^m}{a^n}$ Where a, m, and n are like the numbers above		**Rule**:

Figure 1. Sample worksheet for developing the rule $\frac{a^m}{a^n} = a^{m-n}$.

The expression	Write the expression in expanded form	The simplified expression
$3^2 * 3^4$	$(3 * 3) * (3 * 3 * 3 * 3)$ Note that these are seven 3's.	3^7
$4^3 * 4^2$		
$5^4 * 5^4$		
$6^5 * 6^1$		
$a^m * a^n$ Where a, m, and n are like the numbers above		**Rule**:

Figure 2. Sample worksheet for developing the rule $a^m * a^n = a^{m+n}$.

The lesson ended up being structured as follows: 1) I presented the problem-which was to derive the rules of exponents. The developed worksheets were passed around and the instructor briefly explained the initial step of the worksheet. The remaining challenge was left to the student. 2) Next, I assigned students into groups and asked them to complete the worksheet that was meant to scaffold them through the derivation of rules of exponents. As they were doing their work, I kept asking the students if they were noticing any relationships/patterns in the computations. Then I asked if we could use one algebraic expression that could describe the patterns. 3) Those expressions became the rules and I asked each group to present to the class what they derived. This presentation required students to explain their work, where they struggled, and how they overcame those struggles.

Immediately after teaching the rules of exponents through DL, I asked the students the following questions in a written survey format: (1) where you able to derive all the rules? (2) which part of the derivation was hard? (3) Did you prefer working in groups or alone? (4) What did you like about today's teaching style? (5) How would you have liked to be taught this lesson? (6) Do you prefer DL or DI?

Students' Voices

Preference: DI Versus DL?

There were 23 students in attendance during the DL lesson. Among the 23, 17 students were able to derive all the rules while six students could only derive some of the rules. In terms of instructional preferences, 14 of the students favored DI, five preferred DL, and two students selected both instructional styles. This contradicted my expectations since the majority of the students were able to derive the rules by using DL but yet they would have preferred to learn this topic with DI. We as educators now have to wonder; why would a student prefer to be told a rule rather than derive it? In fact, this is a common phenomenon in my teaching experience; whenever I encourage my students to derive the rules they usually resist by demanding to be told the rules and move on to applying them in practice problems. Maybe this behavior is due to derivations being cognitively taxing or maybe it is consequent of them being used to receiving the rules in a direct instructional method and an abrupt change in teaching methods is not favored by the students. But this brings another question; at what point in the semester should you attempt a different instructional method? Perhaps for students to truly appreciate and be less resistant to an instructional method they need to be exposed to it from the beginning of the semester. But then again, the answer could also be as simple as that the students actually prefer DI over DL.

Collaboration During the DL Lesson

In the case of collaborative work, 14 students preferred working in groups and six students preferred to work alone. Two students selected both options and the final student stated that "it depends on the problem". It is evident that the majority of students prefer collaborative work but it is the last student that caught my attention. S/he made me wonder, as a teacher, what makes an assignment or problem more appropriate for students to collaborate on than another. From my practice as a teacher, I have always allowed students to work together on problems that are deemed difficult. But then one could argue that there is more utility if a student conquers a difficult problem on her own. These are the concerns I reflect on as I read the students' preferences on instruction.

The Struggle to Move from Concrete to Abstract

On this lesson where I decided to use DL for the first time, eight students expressed that coming up with the rules of exponents was not difficult at all. One student said that the difficult part was that the rules were starting to look similar to each other. Another student stated that teaching themselves from worksheets was boring and tedious. The remaining 13 students were more direct in identifying were they struggled. Student *X* said, “I understand how to compute the problems, the struggle came from putting that knowledge in formula form”. Her thoughts summarized the others’ struggles especially on the last rule where they had to derive $a^0 = 1$. This struggle to express concrete computations into symbolic notations that would form rules or formulas is a known problem that students encounter. However, with the freedom to derive these rules I noticed something interesting. The students created their own intermediary step between concrete computations and the rules where they described the rules in words before writing them with symbols. For example, *Student Y* computed $(3^2)^4 = (3*3)*(3*3)*(3*3)*(3*3) = 3^8, (4^3)^2 = (4*4*4)*(4*4*4) = 4^6$, …, then in her penultimate step she wrote, “the base stays the same but multiply the exponents”, and concluded by writing $(a^m)^n = a^{m*n}$.

One of the advantages of DL is that the students can be creative in expressing their knowledge. The majority of the students that struggled with the move from concrete to abstract created this intermediary step. Though the occurrence of this step was a fortunate accident, I recommend to my fellow teachers that this step be incorporated in the scaffolding process as students derive rules or formulas. The void between concrete and abstract concepts seems so large for some students. But by encouraging them to describe in words the patterns they see in their computations, this can help to facilitate the jump to symbolic expressions of the rules.

Students’ Criticisms of Discovery Learning

The DL guidelines recommended that students present their findings to the class after the derivations, however, they did not like this aspect of DL. This was no surprise but I thought asking them to present in groups would alleviate their reluctance. Other students' criticism of DL were varied but interesting nonetheless. Three students expressed that they missed being taught by the teacher. I believe what they truly meant was they missed the highly structured approach of DI. It was until one student expressed that he did not like change in teaching styles that brought me back to a point I made earlier; their negative outlook on DL may be due to me introducing it in the middle of the semester. Another student, who happened to be very bright, derived all the rules within five minutes. This student confessed to being bored for the rest of class and that made me curious on how to engage such students while still using DL. With DI, I usually give such students additional or tougher problems that would ensure that they are occupied while the others are catching up.

The remaining comments were based on fears and limitations they saw in DL. Two students feared that other topics cannot be done with DL and were worried that the freedom to discover could have led them to develop misconceptions. *Student Z* said, "…this (DL) could lead to many students learning the material incorrectly and having to go back and teach them with the previous method (DI) anyway. I don't come to class for DL, that is what online classes are for". *Student Q* adds, "It felt slow, like we could have used that time to actually apply these rules to real problems instead". These sentiments of developing misconceptions and time constraints are shared by some well-established researchers of instructional methods. The fact that an adult learner was also able to evaluate DL and make similar remarks should remind us that these students' voices are not without insight.

How Would You Like to be Taught Rules of Exponents?

After witnessing DL together with their familiarity of DI, the students were able to describe an instructional style that they would prefer. More than half of the class were describing DI as their teaching approach. For example, *Student Q* preferred to be taught "…with the rules given to us, then taught how to identify and use them in a problem". Similarly, *Student P* stated, "…first have the professor explain the lecture/topic, and then do example problems until everyone understands, then give the students time to start the homework, and ask questions if we get stuck." All these answers are more depictions of DI than DL, however, four students favored DL with minor exceptions. One of these students explained that the only thing she would change is to remove the presentation portion of DL. The last three students said that their preference highly depends on the topic being taught. These three students enjoyed DL and are rather open minded in exploring other topics with DL before deeming one teaching method is better than another. In fact, majority of the students said there was more enthusiasm in class when learning with DL than they have witnessed with DI.

Conclusion

My journey in assessing students' preferences on instructional methods has been very informative in my own teaching. There are so many takeaways after hearing student's voices in this exploration. I understood that even though they can perform well under one method they can still prefer another method. In general, my students did prefer DI over DL. One of the major concerns for DL was the jump from concrete to abstract concepts. My recommendation to fellow educators is to intervene with an intermediary step where students describe the computations patterns in words before developing the rules in abstract forms. I also saw their reluctance to do presentations and much of the criticism for DL may be due to when it was introduced. After listening to the students and much reflection, I realized that it might be prudent, for us as educators, to introduce a new teaching method in the beginning of the semester rather than in the middle.

Moreover, the students did enjoy the enthusiasm and the upbeat atmosphere that DL fostered. Majority of them also liked group work but we as instructors need to first investigate which problems are appropriate for group work and for individual work. Most importantly, we need to realize that adult learners are very insightful about their own learning and their instructional preferences should be considered by teachers and even curriculum developers.

However informative this experience has been, many questions about students' voices on curriculum still loom. To what extent should a student govern what and how they learn? Are students' instructional preferences in mathematics related to higher performances? How are we going to measure these performances (for example, attendance, ability to communicate mathematically, achievement motivation, or test scores)? All these are tough questions but I wish to see more influence from students' voices in our teaching of mathematics.

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Sharing Professional Vulnerability: Facilitating Reform from within a Professional Learning Community

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Making sense of the reform teaching ideas from my graduate program was both exciting and daunting. I was starting to form my vision for how I could do a better job of teaching my students. I teach at a three-year high school where any student that I work with will likely learn from two other mathematics teachers during their high school experience. Because transforming my practice to be student-centered would make my methods unique within my department, I wondered whether my students would be well served by transitioning to learning in a student-centered classroom for just one year of high school. I also realized, however, that I would not have to consider this challenging question if other teachers at my school were also changing with me. If I allowed my department members to access some of the ideas from my graduate program, and the ideas were valued enough to take hold, I would not have to change on my own. So, I decided to try to facilitate reform teaching from within my department. In this article I describe how this experience played out, with the intention of providing both hope and insight to others who might find themselves in a situation similar to mine. In telling this story I recount an episode during which my team's interest was sparked; as a result, we changed our practice together. I conclude by sharing my understanding of what made my experience successful and how I have refined my method of sharing my professional vulnerability to position me as a learner (Dale & Frye, 2009), leading to the reform process that we participate in currently.

My Story

My plans for departmental reformation were at first unrealistic and naïve. My initial attempts consisted primarily of sharing portions of articles that I found valuable with my department members and then hoping that we would change together using mathematics education research as our guide. I tried this approach a few times by reserving some of our collaborative meeting time to distribute, read, and discuss compelling papers. Although our meetings showed that my teammates were willing to be respectful while I said what I wanted to say, the ideas were met with ambivalence. My peers responded that the writings I shared contained opinions that were interesting, but our conversation ended there; I had not stirred up any need for them to reform their practice. In reflecting on these early experiences, I realized that I was communicating by telling, rather than facilitating an opportunity for my peers to learn. I was too teacher-centered to make anything work (and the irony was not lost on me). After a few attempts, I discarded my plans for a collaborative reform process and started to make preparation to reform my practice alone. I found myself back at the beginning, wondering if changing alone was worthwhile. I decided that I could always just change back or find another teaching position if a solitary student-centered classroom did not fit into my school's culture. Unfortunately, my plan for success in isolation did not go well either because I found I needed people to bounce ideas off of; changing my teacher identity required support, and I decided I only had one option: turn to my department for help.

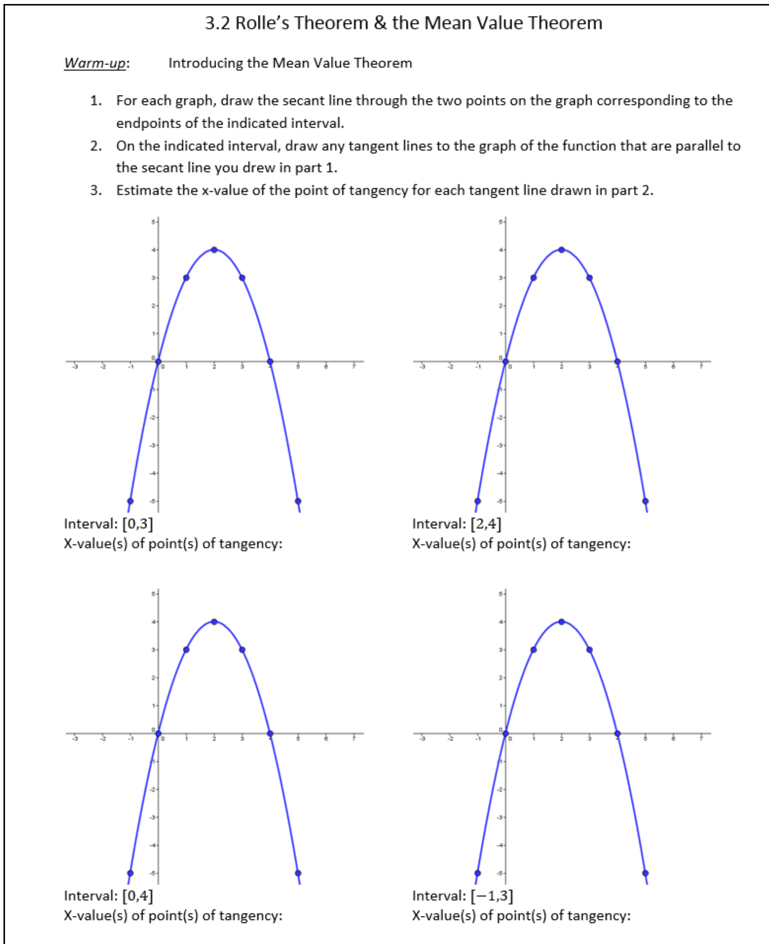
One area where I wanted help was in managing the information that emerged in my class from my student-centered lessons. My students did not seem to be appropriately constructing all of the curriculum that I was supposed to be guaranteeing. My classroom's boards contained student conjectures that I had not anticipated, and I needed help to understand how to modify my student-centered lesson to account for these unanticipated lines of reasoning. When I told my peers that I needed their help in making sense of my students' thinking, they revealed their true nature as supportive teammates that wanted to help me during my time of struggle.

A Calculus Episode

The calculus teachers in my PLC (John, Ashley, and me) have used a common Mean Value Theorem (MVT) lesson for a few years that modifies a resource we got from an AP conference. The episode I describe here took place on a day when I had taught this shared lesson. I had given my students graphs of functions with specified intervals and instructions to sketch the secant line segment that connected the endpoints of the interval and then sketch any lines tangent to the function on that interval that had the same slope as the secant segment. My students' descriptive writing, which was heavily edited with insertions and strike-throughs, was still on my classroom white boards when I met with John and Ashley in my room after school. The student work, paired with the question, "What was the common initial misconception?" was how I wanted to begin the conversation with my team.

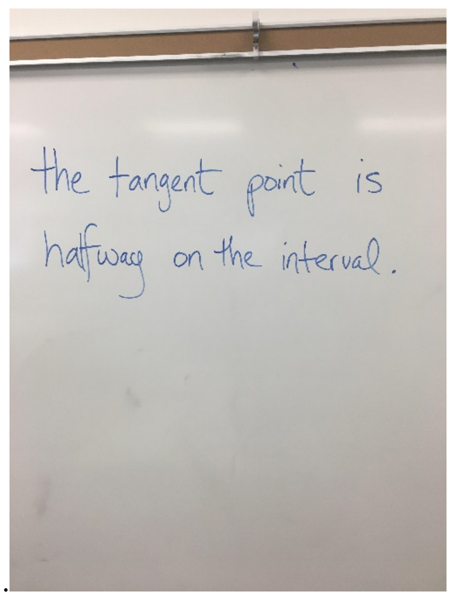
I felt like my students had a misconception that had emerged during the beginning of our lesson, and that all of the groups initially considered this misconception to be foundational to the idea we were discussing. Sharing that my entire class had begun by forming a misconception that I had not anticipated made me feel vulnerable because, although student misconceptions do not always imply a lack of teacher understanding, an entire class of students with the same misconception was enough to make me suspicious of my own knowledge. I recognized the need to modify my lesson and wanted to bring my peers into the conversation.

John observed that a lot of "groups seemed to open with a midpoint idea," (see Figure 1b) in response to the first page of the lesson (see Figure 1a), which made him wonder, "What the crap?" He asked, "Did all of the graphs have the [tangent] point happen halfway through the interval?" I shared that every graph on the first page of the exploration was the same parabola but with different intervals selected. I then claimed that all of the secant segments formed by connecting endpoints of an interval of a parabola would be parallel to the tangent line at the midpoint of that interval. Ashley voiced that it was a pretty cool thing to discover about parabolas, even if it didn't result in the MVT. I agreed and then said that it was so cool that some of the groups had a hard time letting go of that cool thing, and that it was the main idea used by groups during the following sets of non-parabolic graphs.



(a)

Original Student Conjecture



(b)

Figure 1. (a) opening task of original MVT lesson; (b) student conjectures from opening task

I recalled a group's conversation about a non-parabolic graph as "well, it doesn't look like the midpoint is where the point of tangency is, but that's probably just an approximation error because we can't see it well." John immediately understood the students' thinking and laughed while yelling, "Nooooo!" Both Ashley and John suggested we select new graphs for the opening page of the exploration.

I wasn't upset that my students had used the pattern in front of them; I was embarrassed that I had bumped them into an idea that had such an obvious pattern that had nothing to do with the concepts of the lesson. My team agreed that the students were using a valid pattern. I continued to self-criticize: "I put that pattern there. I knew that pattern existed, and I put that pattern there and I need to be more careful about that moving forward." John could tell that I was being hard on myself and I believe he tried to make me feel better when he said, "We've used that for years and it is straight out of what AP gave us." His statement is accurate; he and I typed up an electronic copy of the materials from the AP resources and made Geogebra models to show the same graphics (see file: "MVT lesson old.docx"). I agreed that the materials came from AP and said, "maybe their bad too, but it happened in my class, so my bad." I demonstrated this vulnerability because I did not want to shuffle blame for a misconception that I enabled onto anyone else. I also wanted to own the power to recognize my mistakes and then do a better job. I wanted to share the struggle of reforming my practice and acknowledge that I was having this struggle and that the struggle was a process that I wanted to participate in.

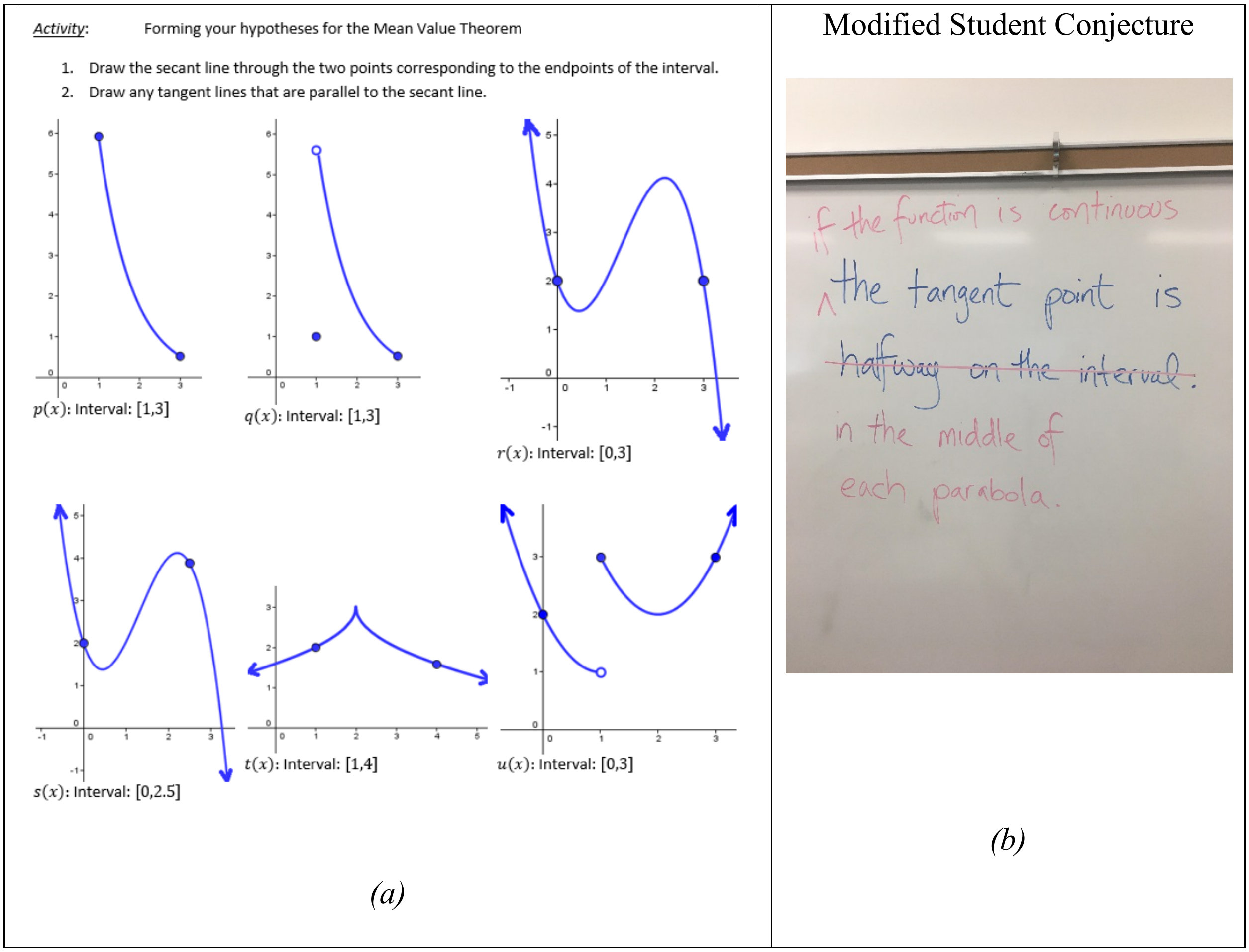


Figure 2. (a) second task of original MVT lesson; (b) student conjectures modified from second task

As the conversation continued, I shared what a student group came up with during class while looking at a graph from the second page of the activity (see Figure 2)—a nonmonotonic cubic function with an interval wide enough to have two tangent lines that match the slope of the secant segment (see file: "MVT student misconception.ggb"). A student reasoned that the graph was really a parabola that stopped and became a second parabola and so the mean of the input values of one end point of the secant segment and the point where the parabola became another parabola would be the same value where the tangent line would intersect the graph. I shared my students' writing with my PLC and waited for their response. My PLC audibly huffed, groaned, and winced at what they saw and then queried about whether what they saw continued with the other graphs. I confirmed that my students were using a process of splitting graphs into pseudo-parabolas. John wondered where they came up with this idea. I said that I believe that it was because of the graphs that I had picked. I described to my PLC how the students considered a natural extension: using parabolic reasoning with means to predict values of functions that are "close enough" to parabolas. My PLC decided we could use some graphs that do not have tangent lines exactly half way through an interval. I suggested using graphs that have a slant asymptote to "lop-side the bumpiness" so that the parabolic pattern wouldn't emerge as the main idea.

We went on to change the first set of graphs by including functions that are not symmetric about a vertical line (see file: "MVT lesson new.docx"). We also discussed the wording of MVT questions on AP assessments. We joked about how MVT should be mVT so it would focus more on slope, which is usually symbolized with lowercase *m*, rather than calculating means. We spoke for a few minutes about math theorem vocabulary and the impact it has on student understanding.

Reflecting on the Episode

Earlier I had tried to share with my teammates an article about student-centered orchestration and it had held no sway over my teammates. Now, however, through sharing my need to reform my practice, that same article became useful in a different manner. My team wanted to understand where my students' reasoning came from and, because of that, wanted to understand how my class was run that morning. Upon reflection, I realized that what seemed to be working was to pair an article or idea about reform teaching with my sharing some aspect of my teaching that required support. I call this approach “Sharing my professional vulnerability,” and this method is changing my department.

My attempts to change my teaching practice typically center around me trying to make use of an idea from research that I found interesting and potentially valuable. The process of improving my teaching practice using mathematics education research is complicated and I often struggle to implement research ideas successfully. Sharing my professional vulnerability is a scenario in which I express my need for help. Some activities that position me as vulnerable include calling on members of my PLC to help me make sense of student reasoning that I did not anticipate, expressing that a recently enacted lesson did not achieve its learning goals, or looking for feedback regarding how I adapted a lesson to try to align it better with mathematics education research. Sharing my professional vulnerability works to mobilize my fellow mathematics teachers to help me understand and make use of a research idea in my classroom. When I initially shared ideas about student-centered learning, other teachers caught on to what I was saying, said that it sounded cool, and then that was the end of our conversation. I think this was because, without sharing my vulnerability, I did not create a need for any action from my peers; I expected something magical to coalescence, which was naivety on my part.

Within the MVT episode above, I shared student writing and my questions about their conjectures. This approach worked well because my practice positioned me as vulnerable, in the form of seeking assistance from my team to understand my students’ thinking. When we met, we quickly focused on finding student misconceptions by reading initial student conjectures and trying to unpack their thinking. The members of my PLC each perceived the same root of the misconception: not all functions are piecewise quadratic; then, we tried to determine why this misconception, which was new to each of us, was emerging from multiple groups of students during my lesson. The main related idea from a graduate course was that monitoring for evidence of student knowledge can be done in several ways, and that selecting student writing was a valuable method for looking at student knowledge (Stein, Engle, Smith, & Hughes, 2008). Every teacher was completely engaged in trying to decide what students were thinking. This is an example of my PLC offering zero resistance to an idea from my graduate program.

When I approached my department and asked them to look at my students' work to help me make sense of what had emerged from the small group discussions, we were engaged in a conversation that had a purpose: helping me to help my students. This conversation was different from what I had initially tried, which had had the sole purpose of increasing teacher awareness of knowledge from research. My peers were trying to make sense of what students had conjectured, how they had formed conjectures, and what I was trying to accomplish by facilitating this interaction. I spoke about how much I must have screwed up because of what the students had postulated. I was honest and vulnerable about the lack of quality of my practice. I was legitimately seeking the assistance of two teachers that would be teaching the same lesson a few weeks later. Suddenly, they were very interested in the details of how I was using small group discussions during class, the same topic that was uneventfully shared three weeks prior.

The other teachers' feedback was useful to me. They pointed out how my lesson plan had nudged students toward a misconception. They also wondered aloud if this thinking had occurred but remained hidden during previous years' instantiations of the same lesson. I shared that I had no way of knowing the ideas that our students had no opportunity to share, and that I was grateful to see their misconceptions so clearly, but also frustrated that I did not anticipate the conjecture fiasco.

Moving forward as a Team

The above calculus episode was the beginning of change for my PLC. Together we revised that lesson, creating a more appropriate sequence of graphs, and setting up conversations between groups of students rather than a lecture; our next few collaborative calculus lessons took on a similar shape. I believe this shift in approach was due to my PLC having valued the experience of reading student conjectures. After a few additional lessons were planned (and a few more episodes wherein my vulnerability to my peers was met with willingness to help me change), we negotiated the details of forming templates for student-centered lesson plans. Since then, we have written dozens of lessons together that focus on how groups of students will make sense of mathematics and share their understanding.

I did not anticipate the impact that sharing my professional vulnerability would have on my team, nor did I anticipate that I would continue this process within my PLC on a permanent basis; I just wanted to start changing my practice collaboratively. I close this article by sharing two experiences that illustrate this impact.

When my team began the most recent school year together, Ashley shared with me the logo that she had found at www.youcubed.org. She told me that she wanted to try and update how she assessed student learning in her classroom based on the website's ideas. She said that she wanted someone to try it out with her and asked if I'd be willing to use the same type of assessments with a class of my students. Although I did not realize it in the moment, I realized later that Ashley had shared with me with an idea from mathematics education research and then made herself vulnerable by expressing her need to not try and change alone. She had employed the very process for improving practice that I have been describing here.

I decided early on to remain transparent with my PLC because I value working with a team that respects one another. Shortly after the calculus episode, I talked with Ashley and John about how my need for help seemed like all the motivation they needed. They were both clear that while they were not entirely comfortable with trying to change, that they wanted to do what is best for student learning and that helping me make sense of student thinking was important. I disclosed that I was trying to get their help with updating my practice and I appreciated anything they could do to assist me to become a better teacher. I also negotiated with them about how much information from my graduate classes I should share so that we got our team talking about student learning quickly.

Sometime later, Ashley, John, and I were chatting in Ashley's room and I described my thoughts about how I wondered whether sharing my professional vulnerability had run its course or whether that process was still in use within our PLC. Ashley said that while she did not think that she was purposefully using her vulnerability to change others, and was not trying to do any research, she had found something valuable and wanted to discuss how she was making sense of it with me. John voiced that he is strictly sharing his needs and that he mostly feels both determined and overwhelmed because he is not sure how to respond to what he sees in his classroom. I said that a large part of what I valued about this approach was that when I was working with them, together as a PLC, they made real contributions to who I was as a teacher. Grappling with ideas that were new to me was easier and more successful for me by sharing the grapple with them. They echoed my comments and I got a sense that we each feel like we are benefiting from our collaboration. We are also each fatigued by this transformational process; the conversation I just shared occurred with Ashley leaning all the way back in her chair, John laying on his side on top of a student table, and me slouched across a few student chairs. We are changing together; we are tired together. It is wonderful.

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Teaching Compound Interest

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A fundamental idea of business mathematics is presented here with a teaching way that tends to bridge the past with the present, high school with college, a method that seems to cover the hiatus that so often educators encounter during their classes. Ergo, the concept of compound interest coalesces to a materialization of a lecture with the employment and substantiation of concepts taught even at the elementary school level, namely fractions and the result of subtraction. Nuances of an everyday classroom lecture are reflected into the comments that this article encompasses.

The choice of an exercise is of special gravity to the understanding of the mathematical material. My audience is first-year college students aspiring to become hotel manages, and as you understand, mathematics can be very easily deemed by them a second-class teaching discipline, since the majority of what they are taught comprises of nonmathematical courses. Students will familiarize themselves with the concept of the compound interest through an exercise that is an amalgamation of this concept and other fundamental ones covered in their past.

Everything I will be writing here revolves around the formula

$$A_t = P(1+r)^t \quad (1)$$

A_t the final capital after t years

P the initial capital

r the annual interest rate

t the number of years

a simplified version of the formula for the compound interest (Compound Interest-MathWareHouse). The reason for doing so is to familiarize the students first with the basic time unit, the year, and later proceed with the usual formula. I will always emphasize at this point the quiddity of interest related problems

$$Compound\ Interest = Final\ Capital - Initial\ Capital$$
$$= P(1+r)^t - P \quad (2)$$

Exercise

An amount is deposited with compound interest into a savings account in the following way: $\frac{2}{3}$ of the amount is deposited annually with annual interest rate 2% for 3 years and the rest is also annually deposited for 2 years with annual interest rate 3%. The difference of the compound interests at the end of each period equals 30\$. Calculate the amount. It is given $1.02^3 = 1.06, 1.03^3 = 1.09, 1.02^2 = 1.04, 1.03^2 = 1.06$ (I use here with enough freedom for the sake of overcoming calculations' obstacles on behalf of the students, the symbol = instead of the typically correct ≅.)

Comment 1 There is mentioning of the concept *fraction* that seems to preoccupy students even in their freshman year. Mathematics is a concatenation of thoughts, an unavoidable reality that our students encounter. So, the exercise provides an incentive to refresh calculations with fractions which will prove to be beneficial to other taught branches of mathematics.

Comment 2 In the given problem, the term *difference* is also present. A concept taught even on the elementary school level. Here it will be employed and substantiated in a rather harder mathematical context, attesting once more to the mathematical concatenation of thoughts.

Comment 3 The choice of the numbers is such that allows the final result to be a natural number. I have experienced during my years of teaching that students feel sort of self-gratification when the result they are looking for is a whole number.

Comment 4 The powers I provide them at the end of the exercise serve a double purpose, first, not to resort to the calculator for calculating them, and secondly, to devote the respective time to deliberating the problem having at the same time a hint and a riddle. A hint, because it sort of guides them what numbers should be employed and a riddle, because they will need to select which one among the four given might lead them to what they are looking for.

I need to emphasize here the gravity of the language while expounding on mathematical concepts. I am under the clout of the sociocultural theory of Vygotsky, where language is a vital component in the process of learning, of cognitive development, "language the very means by which reflection and elaboration of experience takes personal, is a highly personal and at the same time a profoundly social human process" (Vygotsky, 1978, p.126) .

Solution I will split the problem into two subproblems, finding so useful to repeat to myself and to my students a comment by Godin, "Remember that a great strategy in problem solving is to solve an easier problem. This often leads to patterns that we can exploit to solve the original problem", (Godin, June 2017, p.8). I am using the language that I will be trying to apply while teaching the aforementioned exercise.

Let me split the problem the way my initial capital (amount) is split. I have two basic subproblems, calculating the compound interest for the two-thirds and the one-third part respectively. Thus,

$$A_3 = \frac{2}{3}P(1+0.02)^3$$

$$A_3 = \frac{2}{3} P \ 1.06$$

where A_3 the accumulated capital after 3 years and the compound interest according to formula (2)

$$\frac{2}{3} P \ 1.06 - \frac{2}{3} P = \frac{2}{3} P \ 0.06 \quad (3)$$

By the same token, I proceed with the second subproblem, expressing my loyalty to Godin's comment

$$A_2 = \frac{1}{3} P (1 + 0.03)^2$$

$$A_2 = \frac{1}{3} P \ 1.06$$

where A_2 the accumulated capital after 2 years and the compound interest

$$\frac{1}{3} P \ 1.06 - \frac{1}{3} P = \frac{1}{3} P \ 0.06 \quad (4)$$

Comment I attribute special gravity here to the term *equation with one unknown*. The student will realize that what she learned in previous grades is reified in the forthcoming years, and everything has its place in the chain-like appearance of mathematical ideas.

I proceed to the solution of the equation with one unknown by combining (3) and (4)

$$30 = \frac{2}{3} P \ 0.06 - \frac{1}{3} P \ 0.06$$

$$30 = \frac{1}{3} P \ 0.06$$

$$P = \frac{30x3}{0.06}$$

$$P = 1500$$

In the last part of the exercise, the opportunity emerges to deal with fraction calculations and equation solving. What we teachers take for granted, is not in the same way apprised by our students. An exercise like this induces both components of the lesson, students and teachers, to go back to previous grades' mathematical material and bridge potential interruptions in the mathematical concatenation of thoughts. Interruption is also according to Mandler, "Stress-defined as an interruption-can also have the effect of increasing attention to central or crucial events in the environment" (Mandler, 1984, p.253). In my prolegomena, these crucial events bear the names of *fractions* and *equations* and appear to be handled again in a compound interest mathematical setting.

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Summative Assessments for Sixth Grade Mathematics Students: Elementary vs Secondary Schools

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Abstract

In this study the authors conducted an investigative study to determine whether there was a significant difference in the summative end-of-year mathematics assessment scores of 6^{th} grade classes housed in elementary schools vs 6^{th} grade classes housed in secondary schools. The study incorporated a *t* test to compare the mean scores of all the 6^{th} grade classes in Utah. It was found that 6^{th} grade classes who were housed in the elementary schools retained significantly more mathematical knowledge than those housed in the secondary schools.

Acknowledgement

Many thanks are given to Dr. Cora Neal who reviewed the data provided by the Utah State Board of Education concerning the mathematics SAGE scores for 6^{th} graders. She guided the authors in knowing the best tests to run on the data in order to answer the research question.

Literature Review/Background

In the late 1800s there was a push to reorganize schooling from its current practices of an eight-year basic elementary schooling and a four-year high school institution. This grading reorganization was popular because it eliminated the overcrowding of Kindergarten - grade 8 (K-8) elementary students and the high rates of dropouts after grade 8. Intermediate schools or junior high schools soon became part of the reorganization so that by 1920 there were 883 junior high schools in the US and by 1960 four out of five 8th grade students attended a junior high school setting. Junior high schools (grades 7-9) gave way to middle schools (grades 6-8) in the 1950s. This was proposed by Alvin Howard and others to remove the limitations of junior high and allow for the differences in early onset puberty of many students (The Emergence of Middle Schools, n.d.), thus middle schools which included the 6th grade were developed.

In the 1990s as researchers began to look at the elementary to middle school transition, they consistently found there was a reduction of performance (Alspaugh & Harting, 1995). Although many attributed the decline to physiological and psychological factors, Anderman and Midgley (1997) concluded with other similar studies (Eccles & Midgley, 1989; Midgley, Feldlauger, & Eccles, 1989a, 1989b) that the differences between classroom environments due to the transition, "were related to declines in students' expectancies and values in mathematics" (p. 269). Similarly, the transition practices of different schools had little influence on the student scores (Pamperien, 1997).

Surveys during this same 1990s time-period that were given to fifth grade students before entering middle school and to the same students in sixth grade while in middle school, suggested that students felt they were more competent in fifth grade on task goals but in sixth grade where they had more performance goals in the middle school, they felt less competent. This distinction is corroborated in Feldlaufer, Midgley, and Eccles (1988) research, where they found that in the transition year, students "had fewer choices during math, and there was more whole class and less small group work" (p. 293) which are task-focused goal structures that Ames (1990) distinguished from other learning tasks.

Currently in the US, in most districts, the grade levels K-12 (ages 5-18) are physically segregated into three different types of schools: elementary school (or primary school), middle school (or junior high), and high school. For the purposes of this research we consider the State of Utah teacher licensing of K-6 (elementary) and/or 6-12 (secondary) and refer to the schools throughout the state as in the same context of being either elementary or secondary.

Pedagogical Practices between Sixth and Seventh Grades

The State of Utah issues a K-6 teaching license and a 6-12 license. This makes grade 6 very interesting because it is the only grade where both elementary and secondary teachers have access. It may be the case that pedagogical structures are significantly different between elementary and middle school mathematics courses (Jackson & Davis, 2000). The dynamics of the middle school teaching concept of having different teachers for different subjects may make the teacher seem distant or impersonal to the student. Teachers also may have differing licensures that incorporate different pedagogies. Teachers with a K-6 license have more training in cooperative learning strategies than secondary teachers, allowing small group investigations and discussions. While the secondary teachers (6-12 license) are trained more in their specific content area than in how to present the material, therefore they tend to present the material in a lecture format to the whole class. What seems to matter is the structural nature of the secondary school. Jackson and Davis (2000) posited that if teachers collaborate and engage in staff development to facilitate achievement gains with implemented planning times, then scores may differ among schools (Jackson & Davis, 2000).

Most educational settings do not require specific instructional pedagogical practices even though they all require specific instructional content (curriculum). Pedagogical practices are methods, strategies, or styles of instruction. Strategies are selected according to the beliefs of the teacher, the needs of the learner, and the demands of the task (Kervin, Mantei, & Herrington, 2009). Because of this non-specificity, teachers come to the profession or educational context under the theoretical realms of either the banking model (teacher centered) or the comprehensive theoretical pedagogical model (inclusive of all ways of knowing). In order to determine the effectiveness of the teaching and learning, teachers rely on pedagogy and assessments that demonstrate particular types of valued knowledge and show the gaps in that knowledge among students (Popham, 2008, 2010; Black & Wiliam, 2009).

Summative Assessments and Their Influence on Programs and Pedagogy

In 1994, there was a push back, mostly by parents, against the National Council of Teachers of Mathematics (NCTM) because of the shortcomings of the implemented mathematics programs of the 1990s. The programs were failing students in their development of algebra and basic arithmetic skills. A community in Palo Alto, California signed a petition to have the school district keep a traditional pre-algebra curriculum at the middle school. The critique of the current mathematics program was that students' SAT scores in mathematics were diminishing since the implementation of "whole math" in the early 1990s. Similarly, parents in Plano, Texas sued the school district to find another mathematics program than what they were using (Anderman & Midgley, 1997).

When summative assessments or assessments *of* learning take place, the teacher forgoes the right to further instruct the student because the student moves on to a new concept, a different teacher in a different grade, or different class, or even a new school (Stiggins & Chappuis, 2006). For the purpose of this study the summative assessment refers to the end-of-year mandated assessment that is the same for 6th graders whether they are in elementary or secondary schools. The summative assessment is an abbreviated version of what the student has learned over the course of the year. Summative assessments are usually aggregated and provide feedback to teachers and others on what the students in the class, as a whole, on average, learned in the course (Horn, Kane, & Wilson, 2015). Summative assessments traditionally adhere to the banking model theory of teaching as compared to formative assessments that vary in application and in purpose (Freire, 1970).

The onset of state-mandated testing, began a history of using Criterion-Referenced Tests (CRT). Criterion-Referenced Tests assess the knowledge, skills, and abilities of students in the areas of English, Mathematics, and Science, as outlined in the Utah Core Standards. Each individual is compared with a preset standard for acceptable achievement. Student achievement is reported for the individual skills in order to target instruction (Utah State Board of Education, 2016). The CRT was used in Utah from 1999 - 2013. After the implementation of the Utah Core Standards (adapted from the Common Core State Standard initiative) in 2010, Utah participated in the Smarter Balanced Consortium and assisted in the design of computer adaptive assessments. In 2012, Utah State Board of Education chose to withdraw from the Smarter Balanced Consortium to the development and implementation of the assessment called Student Assessment of Growth and Excellence (SAGE). The SAGE end-of-year assessment was completed and implemented in 2014. It is unique to Utah.

Computer-adaptive assessments are a type of technological assessment that is emerging as an alternative to paper and pencil examinations. Computer-adaptive assessments are designed to match the knowledge and ability of a student by adjusting the level of question difficulty, based on the responses delivered by the test taker (Thompson & Weiss, 2011). Adaptive assessments provide the student different test questions according to their performance. The computer-adaptive assessment will randomly present the student with one or two questions. Depending on the answers, the computer will generate the best estimate of the student's ability. After each question, the computer-adaptive assessment will reassess the student's ability and skill level, adapting as needed. The difficulty of the test will adjust to each student's skills, providing a better measure of what each student knows and can do (Hepplestone et al., 2011; Martin & Lazendic, 2017; Thompson & Weiss, 2011). SAGE is also an interactive online assessment (Utah Education Association, n.d.), meaning instead of having all the questions be multiple choice, many questions are written in a manner that requires the students to interact with tools and graphs to discover the solutions.

Research Question

The question for this research in regards to mathematical instruction for 6th graders is: What difference, if any, is there in the mathematics summative SAGE assessment scores of classes that are in 6th grade at an elementary school or 6th grade in a secondary school throughout the state of Utah?

Method

Research (Popkowitz, 1998) has taught us that school spaces are governed by the "discourses that organizes, differentiates, and normalizes the actions of teaching and children" (p 99). School subjects or content areas signify what is to be taught by use of textbooks and required state core curriculum which claim *official knowledge* of the content, and specifically for the purpose of this research, mathematical knowledge. The textbooks and core curriculum provide a preferred logical structuring of the content. As teachers follow the prescribed pedagogical normalization along with associated assessments to determine productivity and competence, the teacher then becomes a tool for disseminating the content in its prescribed formatting. As the teacher embodies the content materials and the formatted pedagogical prescription, the reaffirmation of being successful is corroborated by their work and the students' work. Success is measured not only by final test scores but uniformly matches the disciplining practices of the content through the text, through the pedagogical practices of the teacher, and through the assessments of the individual student. All aspects of the classroom conform to meet the normalizing suggestions of the curriculum and testing to determine if student outcomes are met at the highest possible achievement level. It is assumed that all 6th grade students throughout the state are provided the same mathematics content based upon each school's compliance with the Utah State Core Curriculum Standards for Mathematics (see uen.org for Utah Core Standards, Mathematics Grade 6 Core), yet if scores are significantly different in two spaces, then reasons for the difference could be investigated.

Theoretical Frameworks

A theoretical framework is offered for viewing and analyzing the summative mathematics data collected in order to determine whether there may be differences. In the final discussion the authors offer some possible reasonings for any differences found.

First, the differences in testing scores could be based upon the differing types of surveillance of the teachers on their students in elementary schools vs secondary schools. Foucault (1977) has offered a well-established theory of the "network of gazes" (p. 171) through which we find ourselves throughout various multitudinal spaces in our lives. Institutions are housed and operate through a hierarchy of space and administration that allows for differing types of surveillances of the occupants. Educational institutions are no exception. Elementary schools and secondary schools are structurally different in order to satisfy and maintain different types of pedagogy, surveillance and efficiency. Pedagogy and surveillance are viewed differently in the two diversified spaces. In secondary schools, there are several teachers (up to eight) providing surveillance to different groups of students at differing times. In elementary schools, one to three teachers provide the same surveillance for the same group of students. Because of these differences, students are disciplined differently in the two institutional spaces. Certainly, the disciplining of the body may be different in each space because students move from class to class in secondary schools and maintain the same space for the whole day in elementary school. Also, recess, or unorganized play, is not part of the secondary school day. The disciplining of students may be a factor in the test scores of the students in the two spaces.

In order to determine if the content itself with its accompanying structural pedagogical design is enough to determine student outcome despite the spatial differences of elementary vs secondary school setting, the study was begun. Summative mathematics test scores were gathered of all 6th grade classes throughout Utah. The classes' scores were divided by whether 6th grade classes were housed in an elementary school setting or in a secondary school.

Participants

At the time of the study, the state of Utah had 519 schools that housed 6th grade, 397 elementary schools containing 31,340 students and 122 secondary schools containing 14,574 students, with a total of 45,914 students. The schools used in the study ranged from remote rural schools to urban schools (those schools that are just beginning to experience the challenges associated with urban contexts). Data were gathered from all elementary and secondary public schools that housed a 6th grade class, including all charter schools.

Instrument

To answer the question set forth in the study, the authors used the end-of-year summative assessment scores of all 6th grade classes in the state of Utah for the year 2016-2017. The assessment used by Utah was called Student Assessment of Growth and Excellence (SAGE).

SAGE is a comprehensive computer-adaptive assessment system aligned to the state's core standards (Utah Education Association, n.d.).

In spring 2014, Utah instigated the computer-adaptive, end-of-year assessments in mathematics, language arts, and science for students in grades 3-11. These assessments are aligned to the Utah Core Standards. SAGE is unique to Utah. Hundreds of educators were involved in the development of SAGE (Utah Education Association, n.d.).

Procedure

The Utah State Board of Education (USBE) was contacted to obtain the SAGE mathematics scores for every 6th grade class in the state. USBE required a data release form to be completed and filed with them before they would release the data. The state does not keep records as to whether a 6th grade is in an elementary school or secondary school. However, they were able to provide the grades served in each school. This information was then used as the criteria to determine whether the 6th grade was in an elementary school or a secondary school. If the highest grade served was 6th grade, it was considered an elementary school. After contacting the majority of the school districts, it was determined that if their 6th grade was housed in a school that served higher than 6th grade, then that 6th grade was considered to be in a secondary school.

Once it was determined whether each 6th grade class was in elementary school or secondary school, the means of the elementary 6th grade SAGE mathematics scores were compared with the means of the secondary 6th grade SAGE mathematics scores to determine what the difference might be.

Results

An independent *t* test was conducted to compare students' mathematical achievement for 6^{th} grade classes in elementary school and 6^{th} grade classes in secondary school. The independent *t* test is an inferential statistical test that determines whether there is a statistically significant difference between the means in two unrelated groups. There was a significant difference in the scores for 6^{th} grade classes in elementary school ($M = 413.02$, $SD = 23.48$) and 6^{th} grade classes in secondary school ($M = 403.46$, SD = 23.56); $t(517) = 3.93$, $p < .001$. These results suggest that students in 6^{th} grade classes retain more mathematical knowledge when the grade is housed in elementary school versus secondary school.

It is interesting to report that when looking at the end-of-year summative scores of 6^{th} grade classes in rural schools ($M = 410.80$, $SD = 22.14$) compared to urban schools ($M = 410.77$, $SD = 24.03$) there was no significant difference found; $t(517) = -0.01$, $p = .993$. These results suggest there is no difference in the retention of mathematical knowledge whether a student is taught in a rural school or an urban school. Specifically, that the differences in testing scores are not based upon the spaces in which the students live.

Discussion

Now that a significant difference between the means has been identified and supports the placement of 6^{th} grade in an elementary school setting, the next step is to conduct a study to determine the causes for the difference. When referring back to the theoretical framework that informed the authors on if and why there may be differences in the summative testing scores in elementary school settings and secondary schools, the authors believe that the most likely difference is due to the perceived differences in purposes of the secondary school institution and the elementary school institution. The authors see that the structural differences and the availability of different types of surveillance by only one to three teachers may have a positive effect on students because it allows for a direct connection with what the student is doing over the course of the whole day. In elementary schools, the whole child is disciplined in all aspects of their education not just mathematics and the wholeness supports and controls the students' in all subject areas. Students can be seen as a student of learning rather than a student of a particular content area.

In their intuitive educational beliefs, the authors followed the discourses of the secondary school being a source of more rigor and controlled "business-like" discipline that would allow students greater access to the content and therefore higher scores. The result showed this to be wrong, but the theory was not wrong. We failed to recognize the importance of the discourses of the elementary school as a space for disciplining that may allow for students to learn and engage in the content differently than a business-type model might allow and thus allow more learning. The business-type model is not necessarily more efficient in student mathematics learning with its rigor and discipline as might be supposed. The wholeness discipline found in elementary schools may be the very factor that allowed students to have higher summative testing scores in mathematics and possibly other content areas as well.

The authors found an interesting side note in analyzing the data that suggested there is no significant difference in assessment scores from rural schools to urban schools. This result sparked interest in discovering some theoretical framework in the area of rural school learning versus urban school learning.

Historical discourses would have us believe that student achievement is tied to their living spaces and class distinctions. The authors considered, like Popkowitz (2018) and others (Foucault, 1977; Rose, 1996) in past commentaries, that teachers often embody the pedagogy that maintains distinctions of urban and rural when governing their students through instructional designs of particular content areas. Popkowitz (2018) suggests that urbaness/ruralness is constructed just as much through pedagogy as it is through geographical spaces. He states, "consider the classification of urban and rural as embodying systems of ideas that govern and discipline actions" (p. 22). The normative discourses of lower socioeconomic status tied to ruralness and affluency tied to urbaness and its associated pedagogical practices were not perpetuated by the results of the data found in the current study. The data based upon rural and urban schools found no significant difference. Since rural and urban was not a significant factor, the authors had to look outside of pedagogical practices associated with teaching in rural and urban areas and focus on the differences in pedagogy between elementary and secondary schools. The testing results may match up with the reasons sixth grade was put into a secondary setting in the first place, according to the history, because students in the 21st century are studying algebra at an early age, with more intensity, than they were in the early 20th century. Theorizing that a secondary setting requires more skills may somehow have teachers' pedagogy support that assumption to the detriment of students' actual learning and discipline.

These data from the summative assessments mean that schools, educators, administrators, policymakers, parents, and students can no longer ignore the differences in student achievement at critical transitional stages of education. Students may be unfairly disadvantaged and resources and research need to identify the reasons why and rectify the gaps.

Limitations & Future Research

This study was limited to one year of end-of-year summative assessment data. It would be wise to run the analysis on several years of data. Another limitation with this study is that even though the authors contacted the majority of the schools that serviced grades higher than 6th grade, an assumption was made that any school that serviced grades higher than 6th grade housed the 6th grade in the secondary grades. When in reality it is possible that the 6th grade may have been housed in elementary grades.

It would be profitable to go into the classes of elementary schools and secondary schools during the mathematics instructional time to pinpoint differences in pedagogy but the authors' overall belief is that there are larger concepts of discourse, space, surveillance and pedagogical practices that guide us in our instruction as teachers in the differing spaces of elementary and secondary and therefore become the evidence in the differing assessment results of the students in this study.

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Achieving Equity in the Mathematics Classroom: A Review of The Impact of Identity in K-8 Mathematics: Rethinking Equity-Based Practices

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It is no secret that American students are falling behind, particularly in mathematics. The cause of this achievement gap in comparison to other countries is multifaceted and complex, but much of the burden and blame are unfairly shouldered by those on the front lines of education: teachers. *The Impact of Identity in K-8 Mathematics: Rethinking Equity-Based Practices* (2013) directly addresses those educators. The book manages to circumvent the bureaucracy of American education while raising the level of discourse among educators who desire something better for all of their students, but in particular those who are routinely sidelined in the mathematics classroom. Teachers who read this book are invited to consider the questions, "What mathematics? For whom? For what purposes?" (Aguirre et al. 2013, p. 5), and in doing so will gain a better understanding of the challenges faced by their students with marginalized identities.

The book's authors – Julia Maria Aguirre, Karen Mayfield-Ingram, and Danny Bernard Martin – are all producers of educational research within the framework of equity-based practices. They note their own significant experience in K-8 classrooms as teachers and as teacher educators, as well as experience with parenting their own school-age children in the cases of Aguirre and Mayfield-Ingram. The authors recognize that their individual mathematical autobiographies are replete with examples of positive experiences of mentorship, accelerated academic tracks, and enrichment programs. However, their experiences were also influenced by the fact that their racial, ethnic, and cultural identities come from historically marginalized groups. This combination of experiences and background gives the authors a unique lens and qualified platform from which to write.

In order to accomplish the main goal of the book – namely, helping teachers understand their marginalized students – the authors invite teachers to consider that historically, the questions of "What mathematics? For whom? For what purposes?" (Aguirre et al., 2013, p. 5) have not been given sufficient consideration in curriculum and pedagogy development. Further, teachers are called to consider these questions within the context of their own mathematics identity – that is, "an identity that consists of knowledge and lived experiences, interweaving to inform teaching views, dispositions and practices to help children learn mathematics" (Drake et al., 2001, as cited in Aguirre et al., 2013, p.27). A teacher's formative experiences in the mathematics classroom shape their math identity and can significantly affect how an educator views math and its purpose in real life. Multiple vignettes illustrate the complex process by which teachers come by their mathematics identity and show how students can be "socialized into or out of mathematics" (Aguirre et al., 2013, p. 36) in their K-12 years. Awareness of one's own mathematical history and subsequent ideological perspectives can be a powerful tool in empathizing with marginalized student experience.

One of the greatest strengths of the book is that the authors not only identify and reinforce the problem of inequity in current practices, but they also create a proposed solution in the form of a framework of five equity-based practices: going deep with mathematics, leveraging multiple mathematical competencies, affirming mathematics learners' identities, challenging spaces of marginality, and drawing on multiple resources of knowledge (Aguirre et al., 2013, p.

43). To this end, the book acts as a teacher guidebook. These practices are applied to common issues of lesson planning, assessment, parent engagement, community involvement, and more. The authors are able to effectively show the practicality of applying these ideas in the classroom on a regular basis. They join their voice with others, such as Gutstein and Peterson in *Rethinking Mathematics: Teaching Social Justice by the Numbers* (2013), in showing real-life examples of weaving equity into the core of mathematics lessons, making it seem attainable and even simple.

One of the more powerful accounts is the case study of Mr. C, an urban middle school mathematics teacher, who was alarmed to hear this statement from Joaquin, a student in his class: “This school is always picking on Mexicans” (Aguirre et al., 2013, p. 49). Mr. C took this statement as a call to action and created an immersive mathematics lesson that aimed to examine whether the school indeed “picked on” Mexicans. First, Mr. C made his position on racism absolutely clear by telling his students he refused to work in a racist school and demanded that the injustice, if it existed, must be addressed immediately. He challenged his students to mathematize Joaquin’s statement, and either prove or disprove the claim using real school data relative to suspensions and other types of school discipline. Mathematical concepts like ratios, percentages, and statistics were called into play. Students of varying math ability were able to contribute in meaningful and diverse ways, and they showed higher levels of enthusiasm, persistence, and positivity (Aguirre et al., 2013). Ultimately, Mr. C found that the lessons “made students problem solvers and advocates for themselves and others, thus *centering*, rather than marginalizing, them as confident mathematical learners with a purpose” (Aguirre et al., 2013, p. 53). The case study showed a real, practical application of the five equity-based practices, and an instance in which students belonging to a population with less cultural capital in school were empowered.

The results of Mr. C's case study speak volumes in terms of engaging marginalized students in real mathematics that is applicable to *their* life experience. While the book does not mention the results of the data analysis, his students no doubt benefitted from a serious examination of systemic racism in their school. Allowing students agency in numerical problem solving, teaching mathematics at a deeper, more conceptual level, emphasizing students' individual strengths over deficits, and acknowledging and narrowing spaces of marginality, are powerful ways to increase mathematics learning and engagement. Mathematics teachers of all walks of life can benefit from reading this book and will come away with a new toolkit for greater classroom inclusivity. The book is another example of how intentionally equitable pedagogy is an attainable pursuit and gives hope for the potential of a more inclusive and equitable school system in America.

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