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Please visit the NCTM web site at www.nctm.org/meetings for more information.

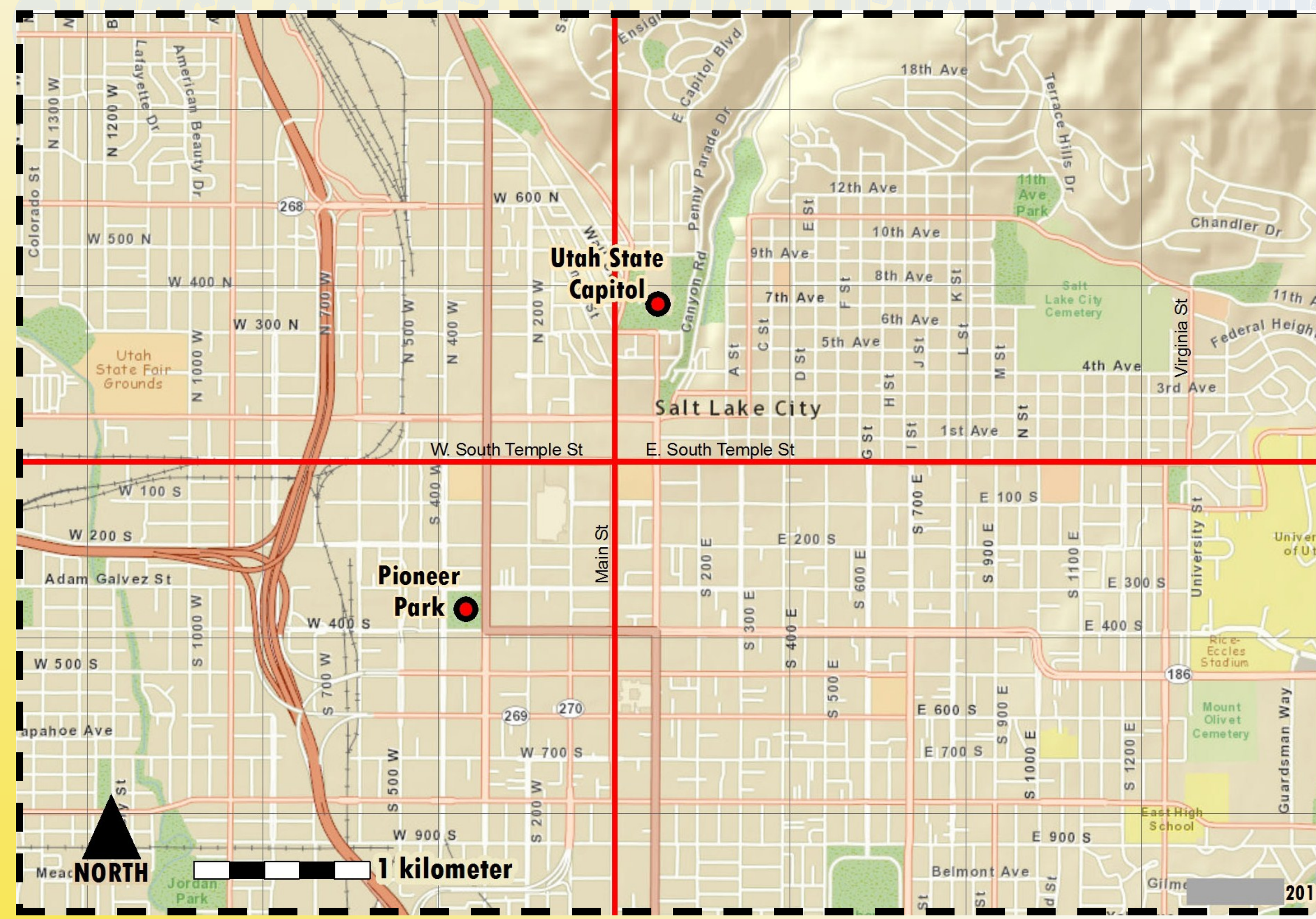
August Problem to Ponder, taken from www.NCTM.org
President's Corner

Students usually know that prime numbers have exactly two factors —1 and the number itself. But, which numbers have exactly three factors? Exactly four factors? Exactly five factors? Exactly six factors?

The general extension: Given a positive integer, n, how can we tell exactly how many factors it has?

Utah Mathematics Teacher
Fall/Winter 2012-2013
Volume 5

Gridded Streets and Rich Historical Origins



http://utahctm.org

Call for Articles

The ***Utah Mathematics Teacher*** seeks articles on issues of interest to mathematics educators, especially K-12 classroom teachers in Utah. All are encouraged to contribute articles and opinions for any section of the journal. Some of the features are: UCTM Leader Spotlight; Letter from the NCTM President; Letter from the UCTM President; Voices from the Classroom; Mathematics for English Language Learners; Puzzle Corner; Recommended Readings and Resources; the Common Core State Standards and Implementation; College and University Research; and others.

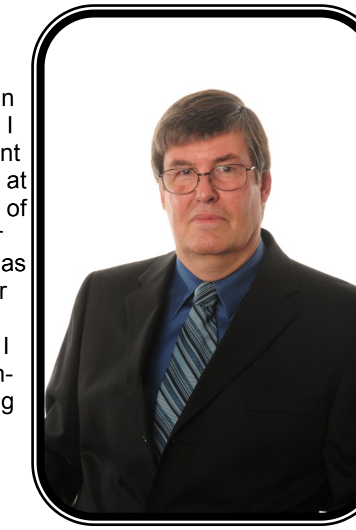
Teachers are especially encouraged to submit articles for ***Voices From the Classroom***, including inspirational stories, exemplary lessons, beginning teacher ideas; or managements tools. Sample ideas are (but not limited to) focused on teachers or districts who have successfully implemented the Common Core State Standards, Inquiry based calculus, and new math programs K-12.

Manuscripts, including tables and figures, should be typed in Microsoft Word and submitted electronically as an e-mail attachment to Christine Walker (Christine.Walker@uvu.edu). A cover letter containing author's name, address, affiliations, phone, e-mail address and the article's intended audience should be included. Items for ***Beehive Math News*** include, but are not limited to, NCTM affiliated group announcements, advertisements of upcoming professional meetings, and member updates.

UCTM Awards '12-13

Don Clark Award

Mike Shaw

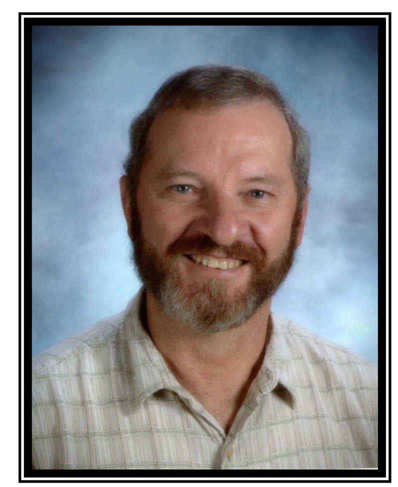


I started my teaching career in 1972 at Churchill Jr. High School. In 1978, I began my high school teaching career at Davis High School, where I remained until my retirement in 2012. I began teaching Advanced Placement Calculus at DHS in 1984, with the goal of making the AP Calculus program at Davis the number one AP Mathematics program in the State. With the help of many others, particularly Gary Taylor, this goal was realized. My reason for wanting DHS to have the number one Calculus program was that I wanted as many Davis High students as possible to have better entry level skills either in college or in career entry programs.

I was the head Cross Country Coach at Davis from 1983 to 1986. I was the head track coach from 1983 to 1985. I gave up coaching to concentrate on raising my family and being able to devote more time to the budding AP Calculus program.

I have two major teaching philosophies:

- **Teaching content is never as important as helping students to shape their thinking skills and their identities**. Luckily, developing students who are better citizens and who are better able to fulfill their goals is not mutually exclusive to teaching content. A statement which I always left on my side board every year was: What you know is somewhat important to becoming "successful." Who you are is FAR more important.
- **Collaboration in teaching is critical to realizing your full potential as a teacher**. Although one can achieve a considerable amount of success working individually, that amount will always be increased if one enlists the ideas and support of their colleagues. I have always attempted to foster collaborative teams when working in the areas of curriculum development, instructional strategies, and development of common assessments. Many other teachers at Davis have been doing the same for at least three decades. Although some fancy names may now be attached to these ideas, the ideas themselves have been around for a long time. What one man can accomplish working alone pales in comparison to what he could accomplish working with others.



Gary Taylor

The single greatest impact on my teaching career was the support and mentoring provided to me as a rookie teacher in 1973 by a "master teacher" who was eager to help me get through the difficult first year.

My main interest in my career has been to seek opportunities to collaborate with other math teachers. I have found everything from a strategy to explain a single math concept to a departmental philosophy is better when accomplished in collaboration. I chose to transfer to Davis High because I knew I would have a chance to work with highly respected math colleagues. I have never been disappointed.

Perhaps the greatest sense of pride I have experienced in 40 years of teaching has come when former students have chosen to become math educators. It is extremely rewarding to have an indirect influence on future math students.

Being an educator for my long career has helped me form an incredible number of wonderful relationships with students and colleagues. Every one of them has enriched my life.

UCTM Awards '12-13

Karl Jones Award

Kinna Harris: After a brief career in commercial health, I decided to go back to school 11 years ago to get a degree in education. I leaned toward becoming an elementary school physical education teacher but it soon became evident that teaching literacy and math was something I was beginning to feel passionate about. The ability to teach kids to think was becoming a new priority for me.

Like most educators, I believe that *all* children can learn. Some catch on quickly- For those kids, I promote deeper thinking. For those who struggle, we celebrate all of the “baby steps”. My students know that it is okay to make a mistake and that it is through perseverance we all learn and grow.

It is my desire to instill a love for math in my students while they are young so they may be successful for years to come. Students feel success when they look at math as a way to organize and understand their world in every day life. Providing multiple experiences for students to become enthusiastic about math through games, exploration, and building a strong foundation of number sense is my overall goal in second grade. Students look forward to math because it is fun.

I have high expectations for my students. I believe they every student can do great things- and because I believe it, they believe it!

George Shell Award

Bonnie Jennings received a Bachelor’s degree from Weber State University in 1975 with a major in Mathematics and a minor in Physics. She received her Masters of Mathematics Education in 2005 from Western Governors University. Bonnie began her teaching career in 1984 where she taught at a private school in Provo. She then taught for five years at the Young Mothers and the Adult High School in the Provo School District. In 1989, she began teaching at American Fork Jr. High in the Alpine School District where she taught for six years and then moved to Timpanogos High School where she is now in her twenty-ninth year of teaching. Bonnie has taught every subject from basic math skills (to 7th graders through adults) through AP Calculus AB. She currently teaches Geometry and Honors Precalculus.

Bonnie has served as the mathematics chair at Timpanogos High since 1996, has served as the new teacher mentor at THS for approximately ten years, and serves on the Alpine district secondary math steering committee. She has been the recipient of the Golden Apple, Crystal Apple, and Special Service awards while teaching at Timpanogos High.

Bonnie married Dean R. Jennings in 1974 and they are the parents of three daughters and one son and the grandparents of one granddaughter and three grandsons. (Who says that the laws of probability don’t hold?)

Bonnie’s teaching and learning philosophy is this: The study of mathematics is both challenging and worthwhile. I enjoy teaching and work hard to create a safe learning environment where all students can enjoy success. Student learning depends upon daily preparation and a willingness to think and reason through difficult situations.

Muffet Reeves Award

Renae Seegmiller*: I have been teaching for 23 years in rural schools including at Piute High School, Laughlin High School in Nevada, and at my current school North Sevier High School. I earned my bachelor's degree from Westminster College, my master's degree from Brigham Young University, and my doctor of education in curriculum and instruction from University of Montana. I have been involved in professional development for the past three years becoming a facilitator for Making Sense of Sense Making and for the Secondary I and Secondary II core academies. I have served on many state committees including the design team for the Secondary II core academy and writing curriculum guides for Secondary I, II, and III.*

Utah Mathematics Teacher

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Presidents Message

Logan T. Toone, UCTM President

As math educators, we all have been involved in conversations regarding Utah's new core standards (based on CCSSO's Common Core State Standards). These standards have served as an overarching topic of our two prior UCTM annual conferences. You will notice this year that the theme of our conference, ***Teach Math Like Your Pants are On Fire!*** is a little different from those of prior years.

Having such a unique and incendiary theme was a conscious decision made with intent to communicate an essential changing aspect of our profession. With new core standards, updated course sequencing, adjusted student placement procedures, and in most schools, newly adopted curriculum, some math educators may feel that the transition to core standards is nearing completion. Such is absolutely not the case! It may be true that the organizational, structural, and curricular transition may be nearing completion, but the instructional transition has only just begun.

In order to maximize the effectiveness of the new learning standards, we need to change the way we teach mathematics. Gone are the days of rote memorization, algorithms for algorithms' sake, and fabricated applications. As we transition to new standards that emphasize critical thinking about number and operations, algebraic reasoning across all grades, and true cross-curricular and real-world application, we need to change the way we teach! If we attempt to teach new standards with old methods, we miss an incredible opportunity. We all know that the single most important factor in the classroom is the quality of instruction we provide, and now that curriculum, courses, and student placement are aligned to new core standards, we must align our instruction as well.

It is my hope that the 2012 UCTM Annual Conference will provide Utah's math educators with ideas, strategies, and reassurance that will support better instruction of new core standards. May we all move forward, responding to the new instructional demands of our core standards and ***Teach Math Like Our Pants are On Fire!***

Have a wonderful year!

Logan T. Toone, PhD
President, UCTM
2010 – 2012

Presidential Award Elementary Finalist

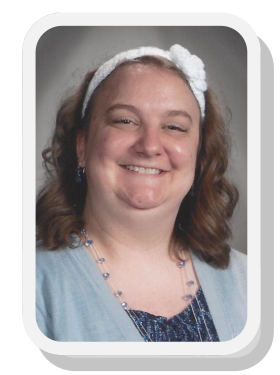
Vivian Shell has taught middle and high school mathematics in the Salt Lake City School District for 15 years. She has spent the past 4 years at the Salt Lake Center for Science Education, teaching and helping build a school for diverse learners. Previously, she taught Prealgebra, Algebra, and English as a Second Language (ESL) Math at Northwest Middle School. During a sabbatical, she worked to help students transition from middle to high school.

Vivian provides all students with access to quality education by working with their individual strengths. In her quest to help students find success and a sense of belonging, she has implemented support structures such as Girls' Math Club, a Math in the City project, Math Lab, and an afterschool homework hall.

Vivian's belief in collaboration has led to work with several learning communities, including cross-curricular teams, networks for inclusive practices, and curriculum design teams. She is a mentor for new teachers in the Support and Mentoring in an Alternative Route to Teaching program with Math for America. She is known for her innovative and accessible teaching methods and has presented throughout the State.

Vivian has a B.S., cum laude, in geology from the University of Utah. She is certified in geology and mathematics education and endorsed in ESL.

Presidential Award Secondary Finalists



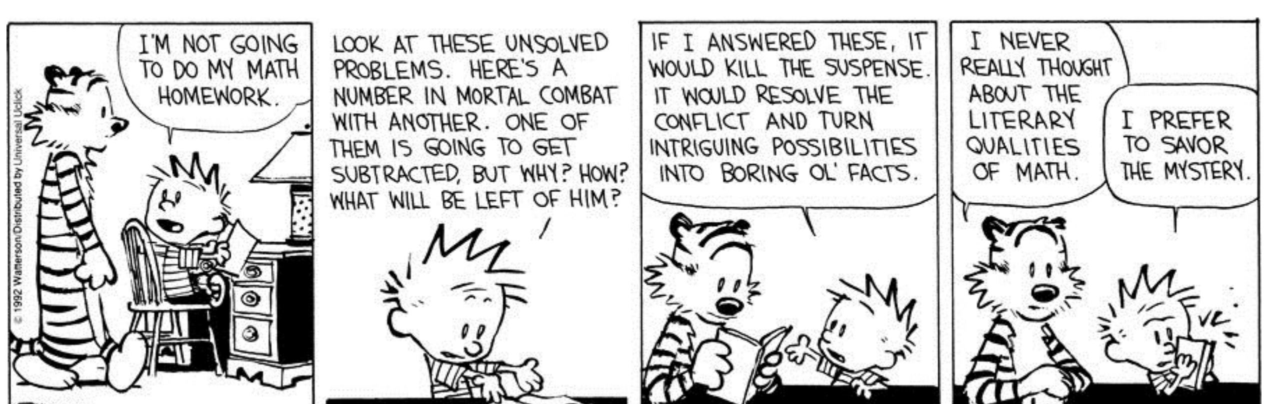
Rebecca Elder received her bachelor's degree and master's degree from the University of Utah. Rebecca has been teaching for 15 years. She started her career by teaching 7th and 8th grade math at Hillcrest Junior High in Murray School District. She then moved to McMillan Elementary in Murray where she taught 5th and 6th grades. She is currently teaching fourth grade. Outside of teaching, Rebecca works for Life Long Learning and Associates which requires her to help school districts in California develop and implement summative assessments in math and science.

Britnie Powell has been teaching for 6 years in the Salt Lake City School District. She has spent the past 3 years teaching 6th grade and Service Learning at the Salt Lake Center for Science Education. Previously, she taught 5th grade at Escalante Elementary School, both are Title I schools. Britnie has a Master's of Education from the University of Utah. She has a Level I Mathematics Endorsement, a Level I Reading Endorsement, and an ESL endorsement. She teaches mathematics, science, social studies, and language arts.

"Ms. Powell is an exceptional teacher, unparalleled in her ability to build a vibrant learning community. She brings a passion for learning, a brilliance and creativity in teaching, and a belief that we can change reality for our students, our school, our community and our world." ~Diane Crim, Utah Teacher of the Year 1999

"...one of the marks of a great teacher is that they realize they don't know everything just because they have a degree in teaching...while the job of a teacher is to help others learn...a great teacher is continually learning."

~McKay Crockett



NCTM NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

To Flip or Not to Flip: That is NOT the Question!

by NCTM President Linda M. Gojak
NCTM *Summing Up*, October 3, 2012

Over the last three decades a variety of instructional strategies have been introduced with a goal of increasing student achievement in mathematics. Such strategies include individualized instruction, cooperative learning, direct instruction, inquiry, scaffolding, computer-assisted instruction, and problem solving. A recent strategy receiving much attention is the "flipped classroom." Innovative use of technology to enhance student learning makes flipping possible and motivating for students and teachers. Simply stated, flipping is a reversed teaching model that delivers instruction, usually at home, through interactive teacher-created videos, while moving "homework" to the classroom. Teachers who have begun to flip their classrooms claim that this approach allows more one-on-one time with each student and increases student motivation, at the same time that students take greater responsibility for their own learning.

I am often asked what I think of the flipped classroom. I do not have any firsthand experience with flipping, so in addition to checking it out online, I have been thinking about it in the context of how mathematics teachers make decisions about what strategies to use when teaching mathematics. What is the potential of the flipped classroom, or any other strategy, to support our decisions on how to teach to ensure student learning?

Teaching is a complex activity. Student needs, teacher content knowledge, conceptual understanding vs. procedural skills, district curriculum, teaching materials, and standards must all be considered as we plan instruction. The Horizon Research Report, *Looking into the Classroom*, concludes that effective lessons are distinguished from ineffective ones by whether they—

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- engage students with the mathematics content;
- create an environment conducive to learning;
- ensure access for all students;
- use questioning to monitor and promote understanding; and
- help students make sense of the mathematics content.

I believe that we need to go further. As we consider effective instruction that leads to student learning, we must remind ourselves of the characteristics of mathematically proficient students. We find these highlighted in the Process Standards identified in*Principles and Standards for School Mathematics* (problem solving, communication, connections, reasoning and proof, and representation), which are the foundation for the Standards for Mathematical Practice in the Common Core, and the strands of mathematical proficiency from the National Research Council's influential publication *Adding It Up*. What is so compelling about the Process Standards is that they provide a critical vision for mathematics instruction and student learning. Rich mathematical tasks provide students with opportunities to engage deeply in mathematics as opposed to a lesson in which the teacher demonstrates and explains a procedure and the student attempts make sense of the teacher's thinking. Communication includes good questions from both teacher and students and discussions that develop in students a deep understanding by wrestling with the mathematical ideas.

Considering some questions about process can be helpful when deciding how you will structure and present a lesson:

- Is this instructional approach appropriate for the grade level of students at this time?
- Can I adapt this strategy so that my lesson incorporates the NCTM Process Standards and encourages students to make sense of the mathematics?

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1) Make sense of problems and persevere in solving them.
2) Reason abstractly and quantitatively.
3) Construct viable arguments and critique the reasoning of others.
4) Model with mathematics.
5) Use appropriate tools strategically.
6) Attend to precision.
7) Look for and make use of structure.
8) Look for and express regularity in repeated reasoning.

Analyzing these Standards, there is *only one* out of the eight that mentions mathematics. When a student peruses the table of contents of their Algebra 2 textbook, it is no wonder they are puzzled as to when they will ever need it. Very few recognize complex polynomial and rational functions in their "everyday life" unless they are a math teacher, a physicist, or an equivalent career...a fraction of working adults. Yet, let's consider the list again. How many of the skills do you apply at any particular job? Any endeavor, personal or professional, makes use of many of the math skills on the Standards list.

Everyone has an idea about how to convey math to others. My goal would be to convey the idea that math is the gym for the mind. It doesn't matter if fifteen years down the road, my students can prove the alternate interior angle theorem or recite Descartes" Rule of Signs. My hope is to pass on an enhanced ability to reason and solve problems. Something that will be valuable to them in any endeavor. That is what math is all about. The mind functions quite like our muscles. Grade school math classes are the weight rooms where we work to develop the mental muscle necessary to go out there on the field of life, and knock down the proverbial linemen that will inevitably stand in our way, regardless of the path that we choose.

When are we ever going to use this?

McKay Crockett, Utah Valley University, Student, Math Education Major

Of all the questions that get asked in math class, I think the most poignant question usually comes from the frustrated kid in the back who raises his hand and inquires, “When are we ever gonna use this?” I've asked this question myself, usually when I am struggling to understand a concept. Though I am still a student, I believe the way a teacher answers this question is indicative of their philosophy on education, math, and learning in general.

I've seen teachers try to satisfy student's inquiries by proposing hypothetical situations where a student might apply a concept in a particular career. This is usually rebuffed by the student when they assert little interest in that particular career. Trying to convince the average student that they are going to be solving cubics, integrating inverse tangent functions, or finding the imaginary roots of polynomials in “every day life” is both a stretch of truth, and an effort in futility. So how does a teacher convey that studying math is important?

Answering the question, “when are we going to use this,” might pose a few analogous questions. If you're on the football team why do you spend hours and hours lifting weights? You're not going to need to bench-press anybody during the game. Why run through tires during practice when there are no tires on the field during a real game? Maybe the student likes music. In that case, why does anybody spend all that time practicing scales and arpeggios? You're not going to get up in the talent show and just play the G-major scale thirty times, right?

Let’s consider the *Standards for Mathematical Practice...*most teachers are familiar with them, yet the key to success in applying the Standards is ensuring that the *students* understand them as well. They are the answer to why we teach math.

- Does this lesson build from a rich mathematical task?
- What questions can I ask students that will encourage them to think more deeply about the mathematics that I want them to understand?
- How can I encourage rich discussions with and among students as they develop understanding and apply the mathematical ideas in a variety of contexts?
- Will my instruction help students to reason and make sense of the mathematics in the lesson?
- In what ways do I anticipate students will represent their thinking about the mathematics?
- How does the mathematics in this lesson connect to previous concepts as well as future concepts?

Although the flipped classroom may be promising, the question is not whether to flip, but rather how to apply the elements of effective instruction to teach students both deep conceptual understanding and procedural fluency. Flipped lessons that simply demonstrate how to do a procedure do not encourage understanding, do not ensure that students will remember the procedure, and do not promote adaptive reasoning. A single instructional approach is unlikely to have a major impact on student achievement once the novelty wears off. A combination of well-thought-out strategies that consider student needs, incorporate the characteristics of effective instruction, and develop understanding of mathematical concepts will have the greatest impact on student achievement.

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Mission & Vision

Mission Statement

A **mission statement** encapsulates an organization's purpose and communicates its essence to members, stakeholders, and the public. It states why the organization exists, what it seeks to accomplish, what it does to achieve this end, and the ultimate result of its work.

NCTM Mission Statement

The National Council of Teachers of Mathematics is the public voice of mathematics education, supporting teachers to ensure equitable mathematics learning of the highest quality for all students through vision, leadership, professional development, and research.

(Approved by the NCTM Board of Directors, October 20, 2012)

Vision Statement

A **vision statement** is a guiding image of an organization's success and the resulting contribution to society. A vision statement describes the best possible outcome and what the future consequently looks like. The purpose of a vision statement is to inspire, energize, motivate, and stimulate creativity.

NCTM Vision Statement

The National Council of Teachers of Mathematics is the global leader and foremost authority in mathematics education, ensuring that all students have access to the highest quality mathematics teaching and learning. We envision a world where everyone is enthused about mathematics, sees the value and beauty of mathematics, and is empowered by the opportunities mathematics affords.

(Approved by the NCTM Board of Directors, October 20, 2012)

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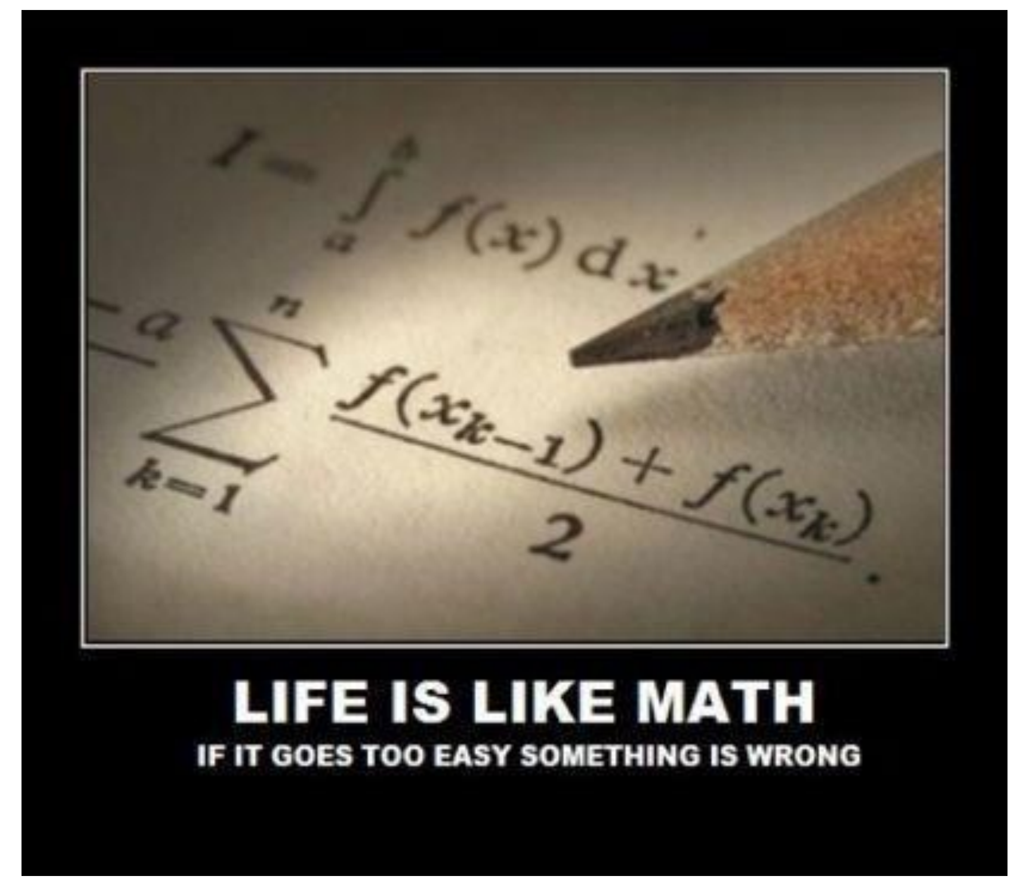
Tinto, V. (1997). Classroom as communities: Exploring the educational character of student persistence. Journal of Higher Education, 68, 599_623.

Trees, A. R., & Jackson, M. H. (2007). The learning environment in clicker classrooms: student processes of learning and involvement in large university-level courses using student response systems. *Learning, Media & Technology*, *32*(1), 21-40. doi:10.1080/17439880601141179

Wade, R. (1994). Teacher education students' views on class discussion: Implications for fostering critical reflection. Teaching and Teacher Education, 10, 231–243.



It is also important that teachers offer precise instructions (Kerssen, 2003). Course requirements must first be presented clearly to students before students can effectively prepare for class. Sidelinger (2010) found that when teachers are explicit in how students can succeed, students will be more likely to prepare for class.

References

Astin, A. (1999). Student involvement: A developmental theory for higher education. Journal of College Student Development, 40, 518_529.

Kerssen-Griep, J., Hess, J. A., & Trees, A. R. (2003). Sustaining the Desire to Learn: Dimensions of Perceived Instructional Facework Related to Student Involvement and Motivation to Learn. *Western Journal Of Communication*, *67*(4), 357-381.

King, P. M. (1978). William Perry's theory of intellectual and ethical development. In L. Knefelkamp, C. Widick, & C. A. Parker (Eds.), New directions for student services: Applying new developmental findings (Vol. 4, pp. 35–51). San Francisco, CA: Jossey-Bass.

Myers, S. A., Horan, S. M., Kennedy-Lightsey, C. D., Madlock, P. E., Sidelinger, R. J., Byrnes, K., & ... Mansson, D. H. (2009). The Relationship Between College Students' Self-Reports of Class Participation and Perceived Teacher Impressions. *Communication Research Reports*, *26*(2), 123-133. doi:10.1080/08824090902861580

Myers, S. A. (2010). Using the Perry Scheme to Explore College Student Classroom Participation. Communication Research Reports, 27(2), 123-130. doi:10.1080/08824091003738016

Petress, K. (2006). An operational definition of class participation. College Student Journal, 40, 821_823.

Sidelinger, R. J., & Booth-Butterfield, M. (2010). Co-constructing Student Involvement: An Examination of Teacher Confirmation and Student-to-Student Connectedness in the College Classroom. Communication Education, 59(2), 165-184. doi:10.1080/03634520903390867

Sidelinger, R. J. (2010). College Student Involvement: An Examination of Student Characteristics and Perceived Teacher Communication Behaviors in the Classroom. Communication Studies, 61(1), 87-103. doi:10.1080/10510970903400311

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NCTM Foundational Priorities

- **Access and Equity:** Advance knowledge about, and infuse in every aspect of mathematics education, a culture of equity where everyone has access to and is empowered by the opportunities mathematics affords.
- **Advocacy:** Engage in public and political advocacy to focus policymakers and education decision makers on improving learning and teaching mathematics.
- **Curriculum, instruction, and assessment:** Provide guidance and resources for developing and implementing mathematics curriculum, instruction, and assessment that are coherent, focused, well-articulated, and consistent with research in the field, and focused on increasing student learning.
- **Professional Development:** Provide professional development to all stakeholders to help ensure all students receive the highest quality mathematics education.
- **Equity:** Advance knowledge about, and infuse in every aspect of mathematics education, a culture of equity where everyone is empowered by the opportunities mathematics affords.
- **Research:** Ensure that sound research is integrated into all activities of the Council.
- **Technology:** Promote strategic use of technology to advance mathematical reasoning, sense making, problem solving, and communication.

(Approved by the NCTM Board of Directors, October 20, 2012)

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101 Questions

Dan Meyer, Keynote
Annual UCTM Conference 2012
American Fork High School

Math teachers find it easy to confuse or bore students. It's much harder to perplex them, to put them in a productive state of wanting to know something they don't know. That's the magic hour for math teachers, when students want an explanation or an activity or a tool that will get them an answer to a question they have.

How do we provoke perplexity? Certain photos and videos can be counted on to generate lots of questions that math can then answer. We're developing a website to share and find those photos and videos now at 101questions (101qs.com). You can upload a photo or video and find out if it's perplexing. You can go to the top ten and find photos and videos that have received the most questions. Most importantly, you can show them to your students and help them get answers and learn math in the process.

What's the first question that comes to your mind?

still be felt by students and teacher and this sense of community is a greater predictor of student willingness to comment in class than class size. Sidelinger & Booth (2010) further explained that student-to-teacher connectedness will produce a positive communication climate. This study supports the notion that even in a large classroom setting teachers have opportunities to incorporate the development of meaningful relationships within the classroom.

It is also important to create a classroom environment where students can feel comfortable and confident. One study found that students feel most comfortable and are more willing to participate in class when "they like the topic or subject matter; if they are interested in their classmates' contributions; if they are familiar with their classmates; or if they believe the participation is vital to their learning" (Myers & Horan, 2009). It is important for students to know and feel comfortable with one another so they will feel comfortable in class. Sidelinger & Booth (2010) found that a significant predictor of student participation is student-to-student connectedness. Interestingly, student-to-student connectedness is correlated with students' willingness to answer teacher's questions (Sidelinger & Booth, 2010).

What Can the Teacher Do

Research gives many suggestions for teachers to facilitate a class environment which allows for optimum student involvement. This Literature Review highlighted only a few items that seem relevant to the discussion of student participation. Given that student-to-student connectedness is the greatest indicator for student participation in class, teachers may want to first encourage student-to-student connectedness before teaching the subject material. "Once connectedness is established, other positive instructional outcomes may be facilitated" (Sidelinger & Booth, 2010). Connectedness can be established through group work and study groups. Sidelinger (2010) stated that when students are given a chance to work out of class with peers, students do not stop thinking about the class material. Group work and study groups should be encouraged to facilitate student-to-student connectedness. By taking the time to do this, students will have greater learning outcomes in the future.

Teachers must use a variety of teaching styles and be approachable. Teachers that are perceived to be more approachable are more likely to foster student participation in class and create a positive learning environment. Because teachers cannot motivate students in the long term, it is important that they encourage students to be internally motivated (Sidelinger, 2010).

The Teacher

For many teachers, student participation in class is an important and desired behavior (Rocca, 2001, cited in Myers & Horan, 2009). Sidelinger & Booth (2010) found that when teachers use a variety of teaching styles and techniques to help students, students are more likely to prepare for class. In addition, an interactive learning style is an important key to student participation. It is the teacher's responsibility to help students develop the ability to be proactive. Sidelinger (2010) stated that a teacher needs to encourage students to have an internal academic locus of control...which was found as an indicator of student involvement and success. Therefore we can say with confidence that the teacher's role is to create a positive learning environment where students can feel comfortable. Myers & Horan (2009) found that teachers who are perceived to be more approachable and tolerant of discussion and disagreement can expect to have more student participation in class.

Despite the desire many teachers have for increased student involvement, class participation is something that many teachers have a hard time achieving, especially in the sciences. The National Science Teachers Association reported that in a college setting many teachers would just give up after several futile attempts at encouraging class participation and would return to a lecture style of teaching (NSAT, 2007).

Teachers have a responsibility to facilitate an environment where students can participate and learn the skills necessary for graduation and eventually success in college and in their chosen fields (Tinto, 2007). Teachers have a challenging task at hand and according to recent studies, it does not appear that students make that task easier. Trees (2007) found that many students may not completely understand their role as active participants in the learning process. It would be beneficial to study in more depth what students feel about their role in the learning process.

The Ideal Classroom Environment

Wade (1994, cited in Sidelinger, 2011) stated that the ideal classroom environment happens when "almost all students are engaged and interested, are learning, and listening attentively to their peer's comments and suggestions". A common conception is that smaller classes are more conducive to the ideal classroom environment. In support of this notion, Sidelinger & Booth (2010) found that there was an overall negative relationship between large class size and student participation. However, they found that even in large classes, a sense of community can

Algebraic Explorations Using the City Grid as Coordinate Axes

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As teachers, we strive to help our students see that mathematics is more than a set of definitions, rules, and formulas in a textbook. Often an algebra student's impression of linear equations may be limited to simple procedures such as: finding slope using rise over run, expressing linear equations in $y = mx + b$ form, and locating (x,y) coordinate pairs by 'going right x units then up y units' to plot a point. However, as stated in *Principles and Standards for School Mathematics*, "Whenever possible, the teaching and learning of algebra can and should be integrated with other topics in the curriculum" (NCTM, 2000, p. 223). In this article, we describe how we utilized maps of two cities with gridded streets and rich historical origins, Washington, DC, and Salt Lake City, to study linear equations creating a 'bridge' from symbolic representations of mathematical ideas to a real-world application. Maps naturally lend themselves to discussions of location and distance, providing a ripe opportunity to connect a familiar setting to algebraic ideas. Abstract mathematical concepts literally become more concrete.

Our discussion begins with a brief historical account of the design and mapping of the capital city of the United States. Street addresses of famous buildings then serve to make connections between real-life use of directional suffixes (NW, NE, SW, SE) and mathematical quadrants. Next, we provide activities for students to find locations using maps with and without grids and construct algebraic expressions to model and find intersections of city streets. Finally, we demonstrate how these ideas and activities translate to the street grid of our own state capital, Salt Lake City.

History of the U.S. Capital

In 1791, President George Washington announced the location of what was to become the capital of our young country. Andrew Ellicott, one of the nation's most prominent surveyors of the time, was hired to survey a square with four ten-mile sides for the capital district (see **fig. 1**), to be centrally located within the existing fourteen states. Ellicott's efforts would not have been realized had it not been for the assistance of his friend, Benjamin Banneker, a free black man and self-taught mathematician and astronomer. Mr. Banneker was responsible for making the solar and astronomical observations necessary for determining the latitude and longitude of the survey (Bendini, 1991). A French architect, Pierre L'Enfant, was hired to design the layout of the city within the surveyed square. He envisioned a city of Baroque style that would express stateliness and power by using grand avenues to connect ceremonial plazas.

Quadrants of Washington, DC

The White House may have one the most famous addresses in our country, 1600 Pennsylvania Avenue NW. But what is the significance of "NW?" Like many major cities, we find addresses in Washington, DC followed by the suffixes NW, SW, SE, or NE: the Library of Congress at 101 Independence Avenue SE; the Jefferson Memorial on Ohio Drive SW; Union Station at 50 Massachusetts Avenue NE; and the Federal Bureau of Investigation (FBI) Headquarters at 935 Pennsylvania Avenue NW. If we imagine a set of axes with the y-axis running north-south and the x-axis running east-west, then the resulting quadrants can aptly be referred to as NE, NW, SW, and SE (see **fig. 1**). As we relate these quadrants to the map of Washington, DC, natural and very important questions arise: "What is the specific location of the origin?" and "On which streets do these axes reside within the city?" An examination of a map of Washington, DC (see **fig.2**) will reveal that North Capitol–South Capitol Street divides the city into east and west addresses, while East Capitol Street, extending through the Mall and the Lincoln Memorial divides the city into north and south parts. Students will discover that L'Enfant laid out our capital city such that our Capitol Building would lie at the origin of the address system. Alternatively, students can discover the origin for Washington, DC by using our interactive map [web address

The Student

A student's attitude toward their learning style is an important aspect of their involvement. Sidelinger (2010) found that these attitudes were consistent predictors of involvement both positively and negatively. "Results found students who perceived themselves to have an internal [academic locus of control] were more likely to be involved. Likewise, proactive personality proved to predict students' involvement inside and outside of the classroom" (Sidelinger, 2010). In addition to this, Myers (2009) stated that students' attitudes toward their learning style such as being internally motivated are directly linked to in-class involvement.

In addition to demonstrating positive learning attitudes such as being internally motivated and having an academic locus of control, students also have an important role to play in the classroom. This role differs depending on class type and culture. In some cultures students are discouraged from asking questions and instead encouraged to listen to the teacher and take notes. Some teachers also have different expectations from other teachers and therefore expect different types of involvement from students. Regardless of cultural differences and teacher expectations, students should approach learning as an active rather than a passive process and should participate in the best way possible to gain the greatest amount of knowledge and skills. This poses an interesting question: how is a student supposed to know how to participate and be involved in the classroom? How can the student know what is appropriate?

Additionally, each student has a different learning style. Myers (2010) study focused on three of the seven types of students discussed in Perry's Theory of Ethical and Intellectual Development. The three stages of student involvement Myers focused on were Dualist, Multiplist, and Contextual Relativist. Knefelkamp & Cornfeld (1979, cited in Myers, 2010) concluded that "Dualist students want to receive information, Multiplist students want to learn how to think, and Contextual Relativist students want to exercise their ability to think" (Myers, 2010). In addition Myers stated that Dualists prefer lecture classes and Multiplists like working in small groups.

Students who prepare for a class report are also more likely to make comments in class (Sidelinger, 2010). It is important that students understand that they must prepare for class in order to have a positive learning experience. Additionally, students must regularly and actively participate in the class discussion to enhance their learning experience (Petress, 2006).

Student Participation in the Classroom

Tristan Wilkie, Utah Valley University, Student, Math Education Major

Introduction

A topic of interest in the study and the search for understanding in the education field is student participation and involvement in the classroom, particularly in the subject of mathematics. This literature review will research the following two questions: What are some of the important aspects of student participation in the classroom? And, what can teachers do to create a classroom environment that allows for the most student involvement?

The Webster dictionary defines the word "involvement" as *"*to occupy (as oneself) absorbingly; *especially*: to commit (as oneself) emotionally." Astin (1999) defined student involvement as ''the amount of physical and psychological energy that the student devotes to the academic experience'' (p. 518). For this study, we will use these two ideas and say that the ideal involvement for students is to continually listen, ask questions, and become emotionally involved in the subject. Involvement is measured by the amount of energy students devote to the subject. Student involvement can range from passive forms of behavior such as taking notes, listening or doing assigned out of class work, to more direct forms of behavior, including oral contributions such as making comments or answering questions. In-class involvement can also include group work such as giving class presentations or working in groups.

Researchers have found that student involvement is crucial to the learning process. Tinto (1997) found that student involvement will lead to a greater understanding of the subject and the development of skills. It is likely that this is because involved students will generally devote a lot of energy to the subject and spend more time at school. Involved students will also interact regularly with teachers and peers (Astin, 1999).

removed for blind review process]. It is interesting to note that the four resulting quadrants (NE, NW, SW, SE) are not equal in size. A quick look back at **figure 1** will reveal that this is largely due to the fact that the origin for the address system does not coincide with the center of the original square. Furthermore, the portion of the square south of the Potomac River, originally donated by Virginia to the United States government, was retroceded back to Virginia in 1846 by congressional legislation and state referendum and is no longer part of Washington, DC (The Historical Society of Washington, DC, 2012). As a result, the Potomac River now serves as the boundary between Washington, DC and Virginia, and the city is not the diamond shape originally envisioned.

If we now consider that Washington, DC lies within a coordinate plane with a grid whose origin is the U.S. Capitol Building and axes are defined by North Capitol-South Capitol Street and East-Capitol Street extended through the Lincoln Memorial, we have divided the plane into quadrants. Up to this point, we have simply referred to these quadrants as NE, NW, SW, and SE. But, how do they correspond to our mathematical coordinate grid? Presenting students with a map of the city that includes NE, NW, SW, and SE markings juxtaposed by the familiar mathematical coordinate grid (see **fig.2**) encourages students to make the connection that NE is equivalent to the mathematical quadrant I, NW is quadrant II, SW is quadrant III, and SE is quadrant IV. Just as values found in quadrants I and II lie above the x-axis and have positive y-values, locations in NE and NW Washington, DC lie north of East Capitol Street extended and have addresses that include "N" (north) in the suffix. Similarly, values found in quadrants III and IV have negative y-values on the mathematical grid and respectively include "S" within the suffix of their address on the DC map. North Capitol – South Capitol Street corresponds to the y-axis of the mathematical coordinate grid with values to the left having a negative x-value and a "W" in their address suffix, and values to the right having a positive x-value and an "E" in their address suffix. In short, the suffix of a Washington, DC address identifies the location (quadrant) in relation to the U.S. Capitol Building.

We use the streets of Washington, DC as a backdrop to our exercise, in part due to its iconic status in the United States, but also due to Pierre L'Enfant's frequent use of 'diagonal' avenues in his design of the city. These diagonal streets interrupt the normal north-south/east-west street grid, and provide an excellent opportunity for students to formulate linear equations. Despite these advantages, almost any other city or location could be used in these exercises. Many cities in the U.S. were laid out in the township and range system, providing a simple Euclidian one-mile by one-mile grid of city streets. For cities with an irregular layout of streets, it is still possible to overlay your own grid system onto the street network, and then have students create linear equations for any long, straight road.

Using a Map with and without a Grid

We chose to distribute two versions of the Washington, DC map for student use. One had a grid superimposed (see **fig. 3**), and one did not (see **fig. 4**). Both were printed at a scale of 1:100,000 so that a centimeter on our map corresponded to a kilometer on the ground. Although this scale provided a convenient conversion between map distances and ground distances, the scale is arbitrary and does not affect the outcome of the exercise. We have provided printable copies of both maps on our website [web address removed for blind review process], along with an interactive online map that is tailored to this activity and which allows for greater precision in taking measurements. Alternatively, teachers looking for maps printed on a very large scale may choose to obtain poster-size road maps.

Both of the maps that we used included axes printed in red that coincided with the city's address system with the origin at the rotunda of the U.S. Capitol Building. The gridded version of the map allowed students to estimate locations to the nearest half centimeter without using a ruler (see **fig. 3**), while the second version of our map had no grid. Students who used this version of the map needed to measure coordinates with a ruler (see **fig. 4**). Having the map presented via these two representations encouraged students to return to the foundational understanding of a coordinate system where (x,y) coordinates refer to the distances from the x-axis and y-axis, respectively. Teachers may note that an advantage to using the map without a grid is that when students measure distances with a ruler, they can estimate to the nearest millimeter rather than approximating to the nearest half centimeter on the gridded version. Using maps both with grids and with axes alone emphasizes that points on a coordinate grid are based upon distances and are not simply codes such as the ones found on a *Battleship* game or a road atlas.

Voices from the Classroom

"As a former educator at an alternative high school I had the privilege of working with hundreds of talented students. These students came from a variety of cultural, familial, and socioeconomic backgrounds. Although their backgrounds were different, all of these students had one thing in common: the traditional classroom setting was unable to adapt to their individualized learning needs. At an alternative high school, the students were able to receive one-on-one individualized attention, helping them learn the basic math skills needed to graduate high school."

~Trisha Rindlisbacher

"Humor in a lesson is like dressing on a salad. Nobody orders a salad because they love to eat salad dressing...but a salad can be enhanced, and made memorable by the use of a dressing. So, don't make the lesson about the humor...make the lesson about math, but make the lesson memorable by using humor."

~Brent Allred

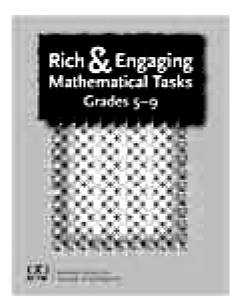
NCTM Recommended Books

Rich and Engaging Mathematical Tasks: Grades 5-9

By Glenda Lappan, Margaret Schwan Smith, Elizabeth Jones

- **Engage your students with rich content in proven mathematical tasks!**
  Promote real understanding of important mathematical concepts
- Design and enact rich instructional experiences that encourage thinking and reasoning
- Lessons follow key areas of mathematics featured in the Common Core State Standards for Mathematics (CCSSM)
  Mathematically rich and engaging tasks offer excellent opportunities for students to learn what mathematics is and how one does it. Such tasks, however, can often be the most difficult to implement effectively during instruction. Research shows that tasks that promote thinking, reasoning, and problem solving often decline during execution due to a variety of classroom factors. The result is students apply previously learned rules and procedures without learning the connection to meaning and understanding and opportunities for thinking and reasoning are lost. This book, a collection of carefully selected articles from past issues of NCTM journals, includes activities for teachers to use with their students to promote the understanding of the mathematical content. Articles are arranged into content strands and ordered within a strand to promote the development of important areas of mathematics. These key areas are all highlighted in the Common Core State Standards for Mathematics (CCSSM) as important for students' progress in mathematics. Each section of the book highlights articles on a key area of mathematics featured in the CCSSM:

- **rational numbers**
- **proportional reasoning**
- **numbers**
- **number theory**
- **patterns and functions**
- **linear equations**
- **measurement**
- **geometry**
- **probability and statistics**



Each article in a section includes mathematical exploration on the focus topic to use with students. **Mathematical tasks** that provide rich content and engage students allow them to not only understand what they have learned but remember and apply it as well. A valuable resource to any mathematics teacher, this rich collection of mathematical tasks will enliven students' engagement in mathematical thinking and reasoning and help them succeed in the classroom.

Investigating Lines on the Map

Once students were able to find coordinates, we considered how to write equations that represented streets on our maps. Students visually recognized that streets could be modeled by lines and were quick to offer the algebraic equation for a line as $y = mx + b$. In order to develop a deeper connection between the algebraic and physical representations of the lines, we asked the students to describe the meaning of the variables and constants in the equation in the context of the map. Drawing upon knowledge that (x,y) pairs are points on the line, students were able to surmise that (x,y) pairs on the map would refer to points along a street. As "b" indicates the y-intercept, students determined that it would indicate where the chosen street intersected North Capitol-South Capitol Street. A couple students quickly pointed out that not all streets crossed North Capitol-South Capitol, so extending our streets (lines) was sometimes necessary to graphically visualize the y-intercept. The value of the slope m, specifically its positive or negative sign, gave a sense of directionality to the street.

Determining if and where lines intersect is a concept students often learn procedurally in algebra. We believed that applying this skill within the context of the DC map would strengthen their conceptual understanding. We asked students to determine if Connecticut Avenue intersects New York Avenue and if so, where. Visually, this was an easy task as students used the maps to see that the streets intersected near the White House (see **fig. 3**). Showing this algebraically was a significantly more involved task. Students first had to find the equations of the lines that represented both streets. For illustration purposes, we use the map without a grid in the discussion that follows. First, we marked and measured two points, $(-3.5cm, 3.4cm)$ and $(-5.3cm, 7.3cm)$, along Connecticut Avenue and used the coordinates to calculate the slope, $m = \frac{7.3-(3.4)}{-5.3-(-3.5)} = -2.2$. We calculated the y-intercept by substituting the coordinates of one of the points and solving for *b*.

$y = mx + b$

$3.4 = (-2.2)(-3.5) + b$

$-4.3 = b$

Therefore, Connecticut Avenue can be represented by the equation $y = -2.2x - 4.3$ .We repeated the process for New York Avenue using the points $(2.4cm, 3.0cm)$ and $(-1.5cm, 1.3cm)$, finding a slope of $m = \frac{1.3-3.0}{-1.5-(2.4)} = 0.44$. We then calculated the y-intercept as shown here:

$y = mx + b$

$3.0 = (0.44)(2.4) + b$

$2.0 = b$

Thus the equation for New York Avenue is $y = 0.44x + 2.0$. Alternatively, or as a check, the y-intercepts can be read directly from the map itself.

With the equations of both the lines in slope-intercept form, students can set the equations equal to each other and ask at what value of x these two streets intersect.

$-2.2x - 4.3 = 0.44x + 2.0$

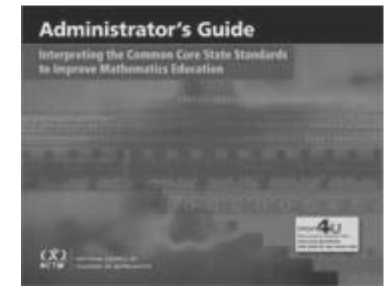
$-2.64x = 6.3$

$x = -2.4$

Substituting this new found value of $x = -2.4cm$ back into either street equation yields the y-coordinate $0.9cm$ and together the (x,y) point, namely $(-2.4cm, 0.9cm)$.

At this point, students can check their answer by seeing where $(-2.4cm, 0.9cm)$ lies on the map. Using the coordinate axes of North Capitol-South Capitol and East Capitol extended, students found that indeed $(-2.4cm, 0.9cm)$ coincided with the White House (see **fig. 4**).

NCTM Recommended Book



Administrator's Guide: Interpreting the Common Core State Standards to Improve Mathematics Education

(Grades K-12)

By Matthew Larson

Nearly a decade ago, NCTM published *Administrator's Guide: How to Support and Improve Mathematics Education in Your School*. This updated *Administrator's Guide* now positions school and district leaders to make sense of the past decade's many recommendations, with special emphasis on the **Common Core State Standards for Mathematics.**

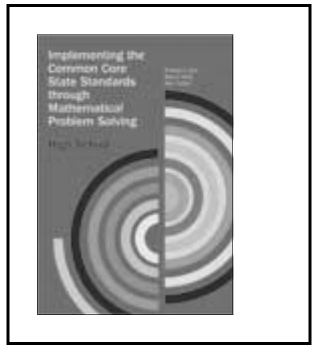
Focusing on similarities between these new standards and those outlined in NCTM's influential *Principles and Standards for School Mathematics*, this handbook efficiently highlights reasoning processes that are essential in any high-quality mathematics program. Students who know these processes are set to "do math" in any context, real or abstract.

Research indicates that many individuals within a school can and do contribute to the work of leading and managing a school. This guide can support anyone who is working to improve mathematics education—either alone or with others.

Make the most of the rare opportunity that the Common Core State Standards offer for rethinking school mathematics and creating exciting new pathways from high school to college and beyond.

Includes answers to "Frequently Asked Questions" and detailed lists of resources to support school mathematics programs.

NCTM Recommended Books



By Frances Curcio, Theresa Gurl, Alice Artzt, Alan Sultan

Connect the Process of Problem Solving with the Content of the Common Core

Mathematics educators have long recognized the importance of helping students to develop problem-solving skills. More recently, they have searched for the best ways to provide their students with the knowledge encompassed in the Common Core State Standards (CCSS). This volume is one in a series from NCTM that equips classroom teachers with targeted, highly effective problems for achieving both goals at once.

The 44 problems and tasks for students in this book are organized into the major areas of the high school Common Core: algebra, functions, geometry, statistics and probability, and number and quantity. Examples of modeling, the other main CCSS area, are incorporated throughout. Every domain that is required of all mathematics students is represented.

For each task, teachers will find a rich, engaging problem or set of problems to use as a lesson starting point. An accompanying discussion ties these tasks to the specific Common Core domains and clusters they help to explore. Follow-up sections highlight the relevant CCSS Standards for Mathematical Practice that students will engage in as they work on these problems.

This book provides high school mathematics teachers with dozens of problems they can use as is, adapt for their classrooms, or be inspired by while creating related problems on other topics. For every mathematics educator, the books in this series will help to illuminate a crucial link between problem solving and the Common Core State Standards.

Applications to Salt Lake City

Like Washington, DC, the city grid of Salt Lake City has a prominent history and thoughtful layout. Brigham Young and his fellow Mormons arrived in the Great Salt Lake Valley in 1847 and within days chose the location for Temple Square, the religious center of Salt Lake City. In just a few years the Avenues neighborhood began to develop with 10-acre square blocks arranged in a grid pattern. Salt Lake City (SLC) enjoys wide streets [132 feet in width], memorialized by Brigham Young's declaration that the streets be "wide enough for a team of four oxen and a covered wagon to turn around" (SLC Corporation, 2005, para. 4). The Avenues are bordered by Canyon Road on the west and Virginia Street on the east, with north-south streets named "A" through "U," and by South Temple Street on the south to 18th Avenue on the north, with streets designated numerically from 1st Avenue to 18th Avenue. Just as the Capitol Building serves as the origin for Washington, DC, the southeast corner of Temple Square at the intersection of Main Street and South Temple Street serves as the origin for the Salt Lake City grid (see **fig. 5**). Newer areas of Salt Lake City lie beyond the Avenues neighborhood where city blocks are about twice the size as those within. In general, these street names consist of three parts: W 400 S indicates that the street runs east-west, lies west of Temple Square, and lies in the 400 block south of Temple Square. Like the nation's capital, the Utah state capital can be divided into quadrants that parallel NE (I), NW (II), SW (III), and SE (IV). Quadrant I, the northeast, is home to most of the historical district. The airport lies in quadrant II to the northwest, while the large majority of Salt Lake City lies in quadrants III and IV to the south.

Unlike the diagonal avenues of Washington, DC, Salt Lake City has few major streets that lie in a direction other than north-south or east-west. Though the SLC map does not readily lend itself to finding equations of lines that represent streets, it does provide opportunities for extension activities to find the distance between prominent places such as the Utah State Capitol Building and Pioneer Park. Students can determine the coordinates of the State Capitol Building and Pioneer Park by using a ruler to measure the perpendicular distance to each axis. The State Capitol is located at $(0.7cm, 2.7cm)$ and Pioneer Park is located at $(-2.6cm, -2.5cm)$. Students then use the distance formula as follows:

$$distance = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(-2.6 - 0.7)^2 + (-2.5 - 2.7)^2}$$
$$= \sqrt{(-3.3)^2 + (-5.2)^2}$$
$$= \sqrt{10.89 + 27.04}$$
$$= \sqrt{37.93}$$
$$= 6.2$$

Therefore, the State Capitol Building and Pioneer Park are $6.2cm$ apart on the map. Teachers should then encourage students to convert this measurement into the real world distance by utilizing the scale bar in the bottom left hand corner of the map. The scale bar indicates that $3cm$ on the map is equivalent to 1 kilometer in the real world (Note: The size of the scale bar on the Salt Lake City map may vary due to printer settings). Since the landmarks are $6.2cm$ apart on the map, they are actually just over 2 kilometers apart [$\frac{6.2\ cm}{1} \times \frac{1\ km}{3\ cm} \cong 2.1\ km$]. Students may prefer measurements in miles and can use the fact that 1 kilometer = 0.621371192 miles to deduce that the State Capitol Building is a walkable 1.25 miles from Pioneer Park!

Conclusion

Most coordinate grids appear simply as black lines over white paper. We have superimposed coordinate grids over colorful maps of Washington, DC and Salt Lake City, transforming discussions of quadrants into that of street prefixes and suffixes, (x,y) coordinate pairs to locations of famous buildings, equations of lines to numerical representations of streets, and numerical solutions of intersections of lines to physical intersections of streets. We hope our application of maps as a means to study linear equations enlivens your classroom discussions and provides a concrete application of fundamental algebraic concepts beyond symbolic manipulation.

What can you conclude?

Conclusion

As pointed out at the beginning of this article Ceva's Theorem generalizes statements that are frequently covered in high school geometry classes, its proof uses geometry concepts traditionally covered in high school and can be investigated using technology. The lesson worksheet provided can be used as an in class activity in the geometry class, as well as an out of class project. The author has tested the lesson worksheet during her outreach program *Math Days for Women*, few years ago. She led approximately 20 high school female students (from five participating high schools) through the steps of the proof of the Ceva's theorem. First, students used technology to discover and explore the statement of the Ceva's theorem and its corollaries. Then, with appropriate guidance and help, students were able to proof the theorem, were impressed how the various concepts learned before helped them to achieve that, and at the same time had fun doing that. Exploring Ceva's Theorem can be an interesting, fun and rewarding experience.

Note: The technology exploration took about 50 minutes (but also included introduction to the software). After the statement of the theorem was discovered, proving the theorem took about 35-40 minutes.

Reference:
1. Michael Hvidsen, *Geometry (with Geometry Explorer)*, McGraw Hill, 2005
2. Martin Isaacs, *Geometry for College Students*, The Brooks/Cole, 2001
3. Gerard A. Venema, *Foundations of Geometry*, Pearson, 2006
4. Gerard A. Venema, *Exploring Advanced Euclidian Geometry with Geometry Scetchpad*, lab manual (online)
5. GeoGebra website (http://www.geogebra.org/)

Then $\frac{area(\Delta ACF)}{area(\Delta BCF)} =$ (2)

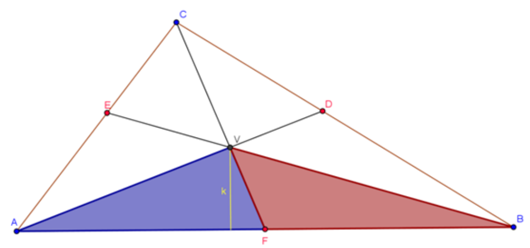


Figure 3

Using Figure 3 we find: $\frac{area(\Delta AVF)}{area(\Delta BVF)} =$ (3)

Denote $\frac{AF}{FB} = r$. Using this notation, the equations (2) and (3) can be rewritten as

$area(\Delta ACF) =$ (4)

$area(\Delta AVF) =$ (5)

Subtract (4) and (5): $area(\Delta ACF) - area(\Delta AVF) =$ (6)

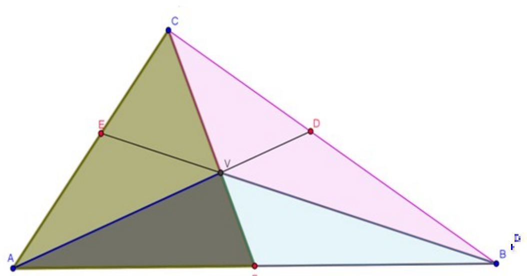


Figure 4

Compare the areas of the triangles that appear in equation (6) [see Figure 4]. Use this observation to rewrite equation (6) using ΔAVC and ΔBVC

$area(\Delta AVC) =$ i.e.

$\frac{area(\Delta AVC)}{area(\Delta BVC)} =$ (7)

Similarly $\frac{area(\Delta AVB)}{area(\Delta AVC)} =$ (8)

$\frac{area(\Delta BVC)}{area(\Delta AVC)} =$ (9)

Multiply the equations (7), (8) *and* (9): $\frac{AF}{BF} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} =$

References

Bendini, Silvio. 1991. The Survey of the Federal Territory: Andrew Ellicott and Benjamin Banneker. Washington History 3 (1): 76-79.

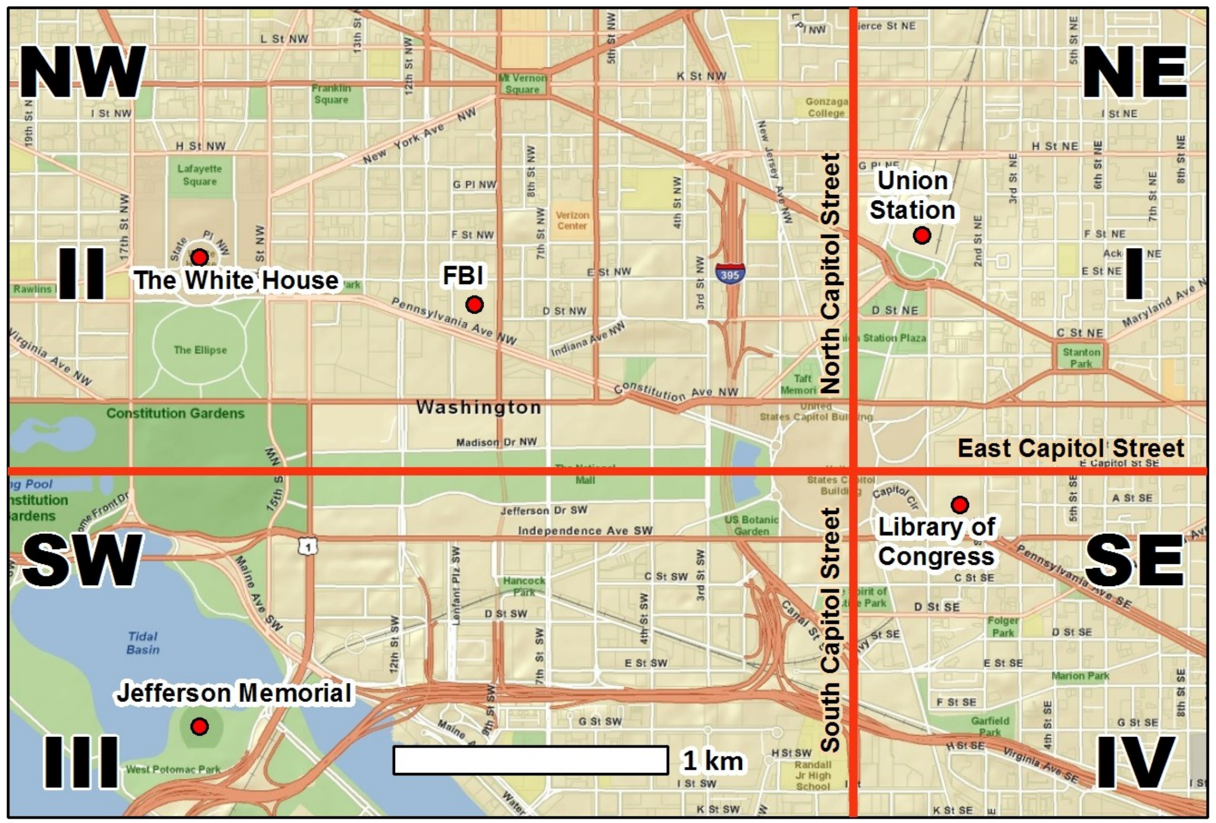
National Council of Teachers of Mathematics (NCTM).*Principles and Standards for School Mathematics*. Reston, VA: NCTM, 2000.

The Historical Society of Washington, DC. (2012). *Get to know DC*. Retrieved from http://www.historydc.org/aboutdc.aspx

Salt Lake City Corporation (2005). *A city takes shape.* Retrieved from http://www.slcclassic.com/info/area_info/salt_lake_city.htm.

Figure 1

Figure 2



(**Note:** The exploration in 9 will show that the only possibilities are either zero or two of the points *D*, *E*, and *F* to lie outside of the triangle ΔABC. This is incorporated in the sensed ratio in the Ceva's Theorem: just two of the ratio can be negative or none of them.)

Lesson Worksheet

The following worksheet can be used to guide students in their discovery of the proof of a version of Ceva's Theorem.

Ceva's Theorem (version of it):

Let ΔABC be a given triangle, and let D, E, and F be nonvertex points that lie on the sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ of the ΔABC respectively. If the lines $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$ are concurrent (intersect at a point) then

$$\frac{AF}{BF} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1. \qquad (1)$$

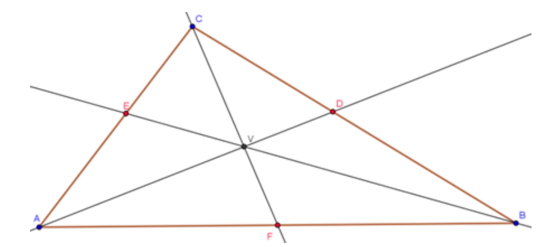


Figure 1

Proof: Let ΔABC be a given triangle, and let *D*, *E*, and *F* be points that lie on the sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ of the ΔABC [Figure 1].

Suppose $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$ intersect at a point *V* as in Figure 2.

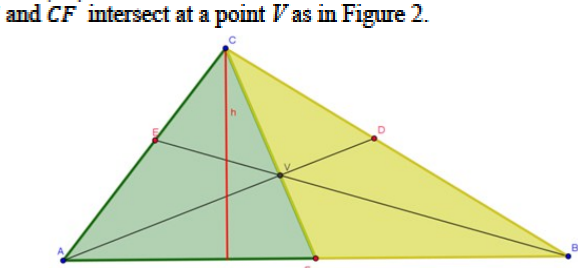


Figure 2

Exploring Ceva's Theorem using GeoGebra:

For centuries, important part of mathematics has been *drawing* and *visualizing*. The *GeoGebra* (*www.geogebra.org*) is designed to help students with math visualization in a practical, fun and engaging way. It is friendly software that can easily be self-taught.

The *GeoGebra* is a "dynamic mathematics software that joins geometry, algebra and calculus. It is developed for learning and teaching mathematics in schools by Markus Hohenwarter and an international team of programmers." In geometry, this software program gives opportunity to explore and discover geometric results by building and investigating mathematical objects, figures, diagrams, and graphs.

In this section we see how this software can be used to explore Ceva's Theorem. The following is one strategy to do that:

1. Use the Polygon tool to create a triangle ΔABC.
2. Use the Line through Two Points tool to create the lines $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$, and $\overleftrightarrow{AB}$.
3. Use New Point tool to create arbitrary points D, E, F on the lines $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$, and $\overleftrightarrow{AB}$, respectively (create the points on these lines but outside of the sides of the triangle).
4. Use Line through Two Points tool to draw the lines $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, $\overleftrightarrow{CF}$.
   (**Note:** Choose the points D, E, and F, such that the lines $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, $\overleftrightarrow{CF}$ do not meet at a point.)
5. Use the *Input Bar* at the bottom of the GeoGebra window to calculate
   a. *p= AffineRatio[A, B, F] / AffineRatio[B, A, F]*
   b. *q= AffineRatio[B, C, D] / AffineRatio[C, B, D]*
   c. *r= AffineRatio[C, A, E] / AffineRatio[A, C, E]*
   d. $s = p\,q\,r$
   (**Note:** The calculated values of p, q, r, and s will appear in the Algebra View.)
6. Is s=1?
7. Try to adjust the points (by moving point D, for example, using the Move tool) so that s is exactly 1.
8. What can you notice about the lines $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$ when s=1?
9. When s=1, move the points A, B, and C, around (using the Move tool) and observe if s changes.
   (**Note:** Compare your discoveries in 7 and 8 with the statement of Ceva's Theorem.)
10. Let V be the point of intersection of lines $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, and $\overleftrightarrow{CF}$.
    a. Move the points A, B, and C, around such that V stays inside the triangle ΔABC? What can you conclude about the points D, E and F? What sign the ratios p, q and r have?
    b. Now, move the points A, B, and C, around such that V moves outside of ΔABC. What can you conclude about the points D, E, and F? What sign the ratios p, q and r have?
    c. Can you find a location of V such that exactly one or three of the points D, E and F, lie(s) outside of the triangle ΔABC?

Figure 3

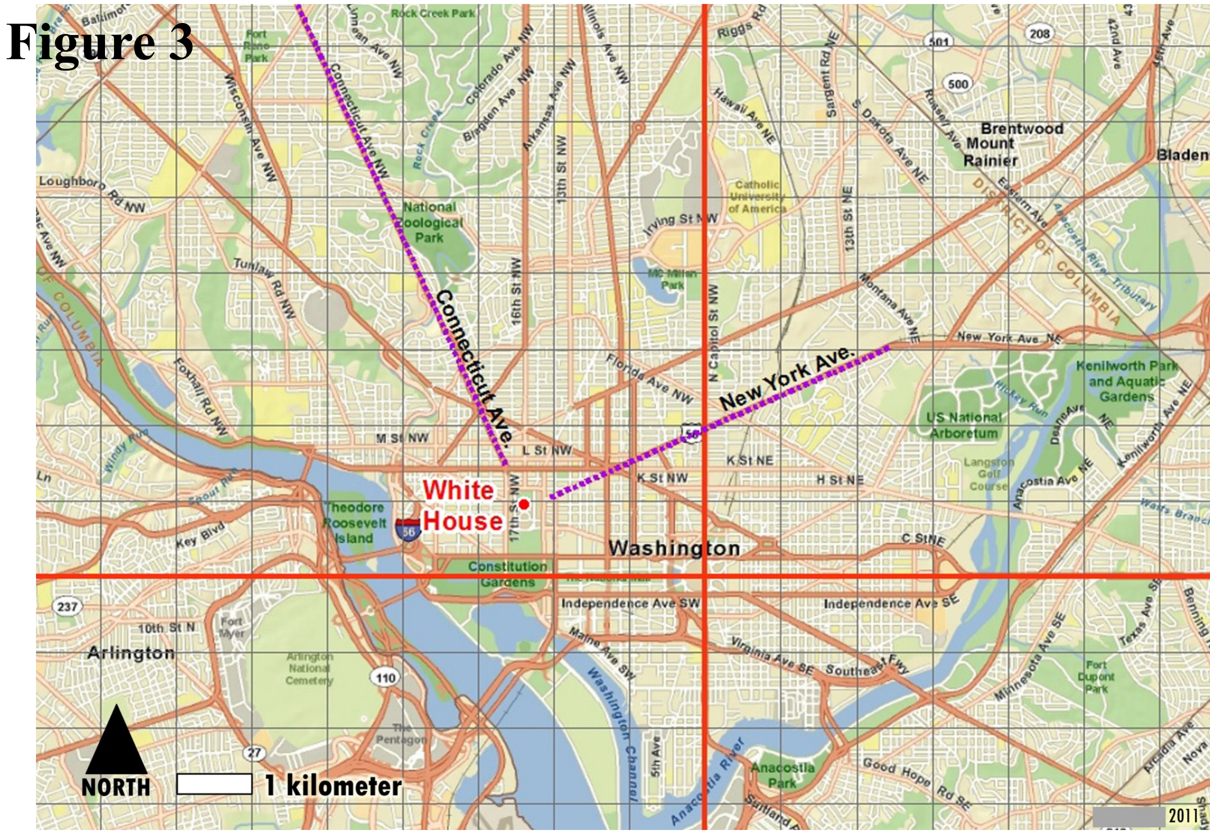


Figure 4

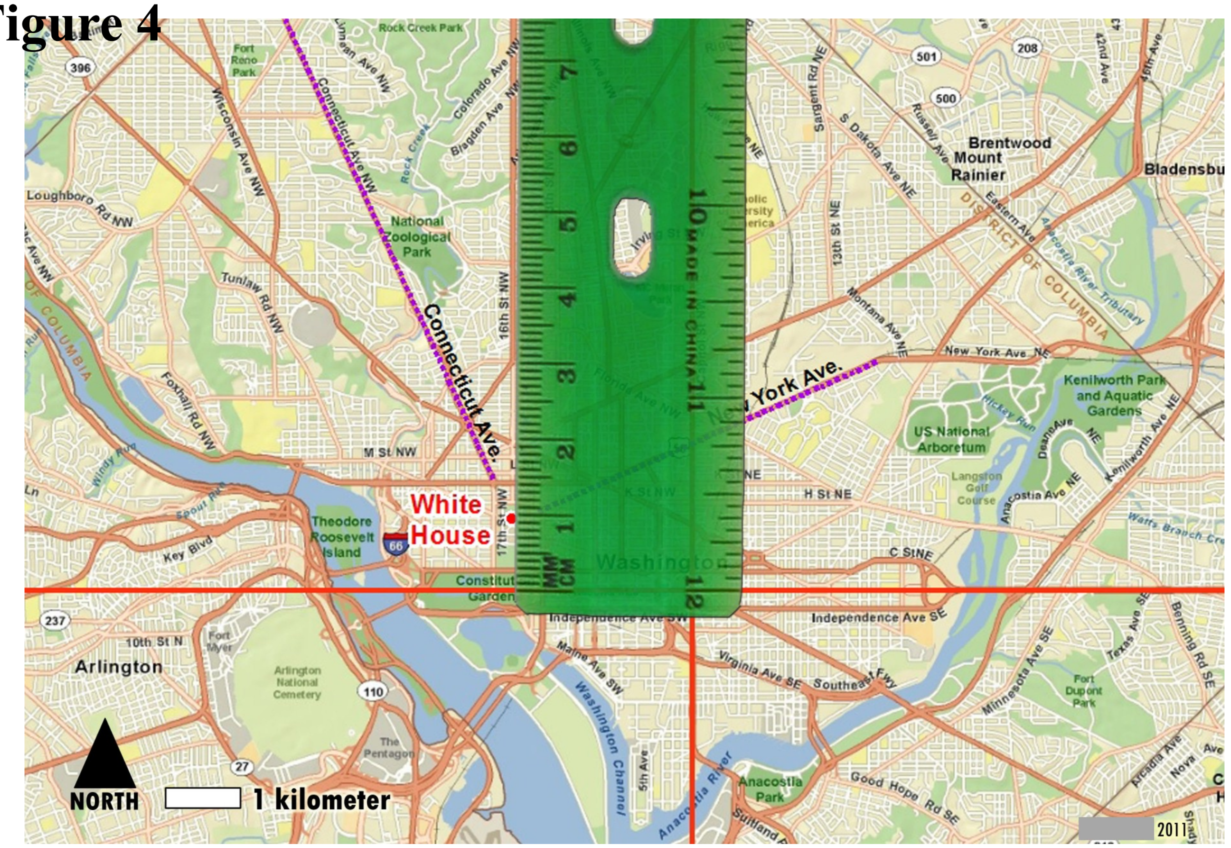
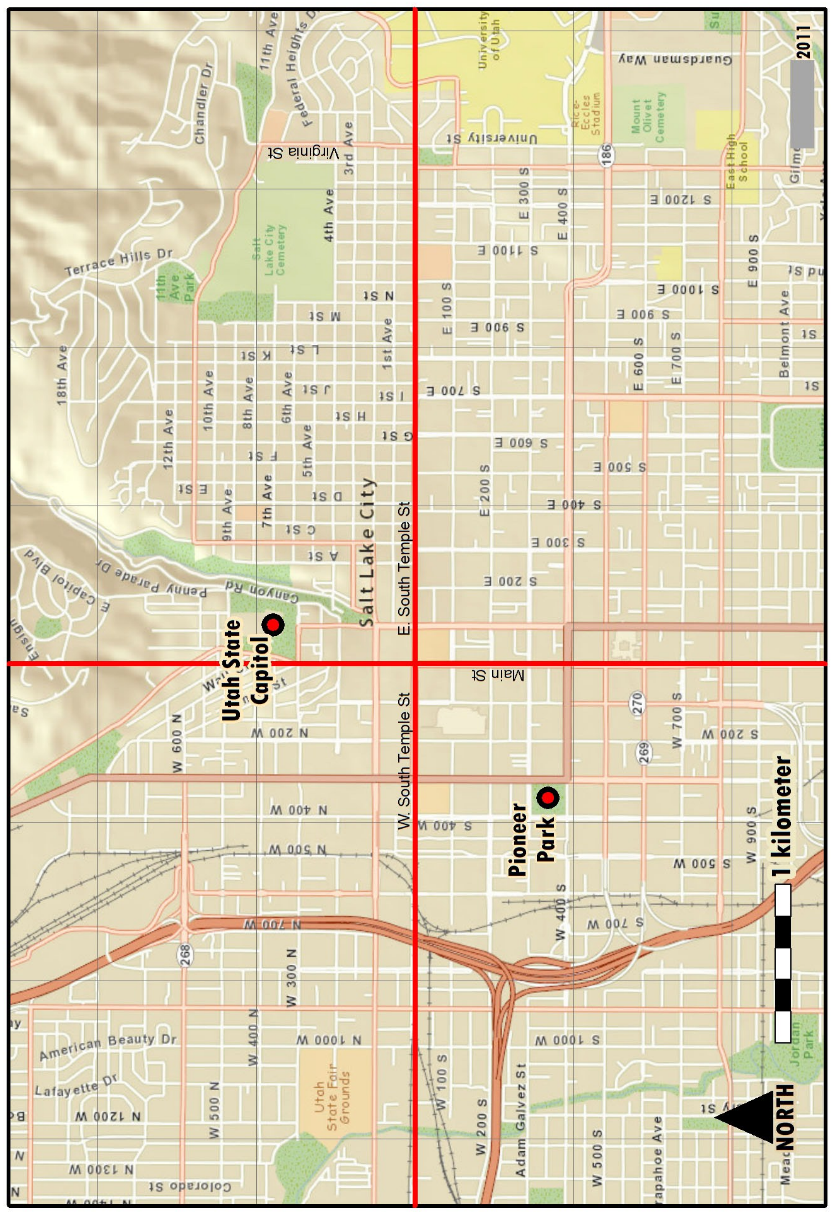


Figure 5



Since $\angle AFC = 180 - \angle BFC$, then using properties of the sine function we have $\sin\angle AFC = \sin(180 - \angle BFC) = \sin\angle BFC$, hence

$$\frac{AF}{CA} = \frac{BF}{BC}, \quad \text{i.e.} \quad \frac{AF}{FB} = \frac{AC}{BC}.$$

Similarly, applying the sine law to the corresponding triangles, we get the other two equations in (11).

$$\frac{AF}{FB} = \frac{AC}{BC}, \quad \frac{BD}{DC} = \frac{BA}{CA}, \quad \text{and} \quad \frac{CE}{EA} = \frac{BC}{AB}. \tag{11}$$

Now, multiplying the three equations in (11) we get:

$$\frac{AF}{BF} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{AC}{BC} \cdot \frac{AB}{AC} \cdot \frac{BC}{AB} = 1.$$

Hence, by the Ceva's Theorem it follows that angle bisectors intersect at a single point. ■

The point where angle bisectors intersect is called the *incenter* of the given triangle (the center of the inscribed circle).

Corollary 3: (Orthocenter) In a triangle, altitudes intersect at a single point.

Proof: Let ΔABC be a given triangle, and let $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ be the altitudes (where D, E and F are the feet of the altitudes that lie on the on the lines $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$, respectively).

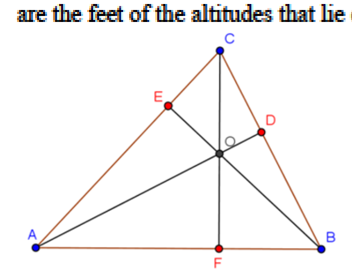


Figure 6

Note that right triangles ΔAFC and ΔAEB are similar by AA Similarity Theorem. Therefore

$$\frac{AF}{EA} = \frac{CF}{BE}.$$

Similarly, $\frac{BD}{FB} = \frac{AD}{CF}$ and $\frac{CE}{DC} = \frac{BE}{AD}$. Then

$$\frac{AF}{BF} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{CF}{BE} \cdot \frac{AD}{CF} \cdot \frac{BE}{AD} = 1.$$

By the Ceva's Theorem it follows that altitudes intersect in a single point. ■

The point where altitudes intersect is called the *orthocenter* of the given triangle.

The Ceva's triangle ΔDEF is called *orthic triangle* for the ΔABC.

Comparing the equations (1) and (10), we have

$$\frac{AG}{GB}=\frac{AF}{FB}.$$

The last equation shows that F and G divide AB in the same ratio, so F and G are the same point, which proves that the lines $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, and $\overleftrightarrow{CF}$ are concurrent.

Applications of Ceva's Theorem

In this section we prove statements that follow directly from the Ceva's Theorem. Most of them are frequently covered and used in high school geometry classes.

Corollary 1: (Centroid) In a triangle, medians intersect at a single point.

Proof: Recall that a *median* is a segment that connects vertex of a triangle with the midpoint of the opposite side.

Therefore, given a triangle ΔABC, with D, E and F midpoints of the sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$, respectively, we have $AF=FB$, $BD=DC$ and $CE=EA$. Then

$$\frac{AF}{BF}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=1\cdot 1\cdot 1=1.$$

Hence, by the Ceva's Theorem it follows that medians intersect in a single point. ■

The point where medians intersect is called the *centroid* of the given triangle.

Corollary 2: (Incenter) In a triangle, angle bisectors intersect at a single point.

Proof: Let ΔABC be a given triangle, and let $\overrightarrow{AD}$, $\overrightarrow{BE}$, and $\overrightarrow{CF}$ be the angle bisectors (where D, E and F are points on the sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$, respectively).

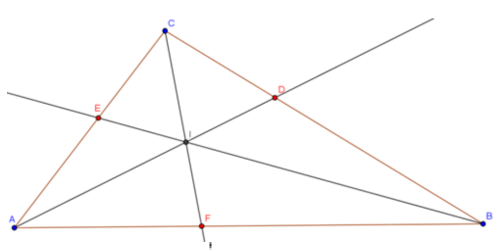


Figure 5

Applying the sine law to the triangles ΔACF and ΔBCF we get:

$$\frac{\sin\angle ACF}{\sin\angle AFC}=\frac{AF}{CA} \quad \text{and} \quad \frac{\sin\angle BCF}{\sin\angle BFC}=\frac{BF}{BC}$$

respectively.

CEVA's Theorem

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Giovanni Ceva [pronounced: *Chay'va*] (1648-1734) was an Italian mathematician, who most of his life worked on Geometry, particularly geometry applied to triangles. In 1678, he discovered and published one of the most important results in Geometry. Today this remarkable result bears his name – Ceva's theorem.

In this article we look at this important theorem and its proof, and discuss its corollaries – results that are frequently used in high school geometry.

Notation: The following notations will be used throughout this article:

- $\overleftrightarrow{AB}$ – line through the points A and B.
- $\overline{AB}$ – line segment with end points A and B.
- $\overrightarrow{AB}$ – ray with end point A.
- AB – length of the segment $\overline{AB}$
- $\frac{AF}{FB}$ – a *sensed ratio* (if F is a point on $\overleftrightarrow{AB}$, such that F is between A and B, then $\frac{AF}{FB}$ is

positive, and otherwise negative). (For example, in Figure 1b, the sensed ratio $\frac{AF}{FB}$ is positive, since F is between A and B, but $\frac{BD}{DC}$ is negative, since D is not between B and C.)

Ceva's Theorem:

Let ΔABC be a given triangle, and let D, E, and F be nonvertex points that lie on lines $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$, and $\overleftrightarrow{AB}$ respectively. Then the lines $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$ are concurrent (intersect at a point) if and only if

$$\frac{AF}{BF}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=1, \tag{1}$$

where $\frac{AF}{BF}, \frac{BD}{DC}, \frac{CE}{EA}$ are sensed ratios.

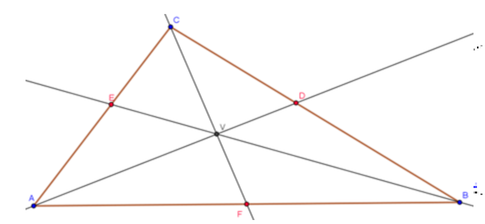

Figure 1a

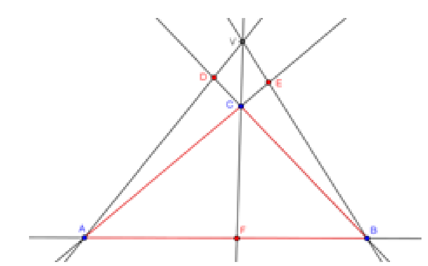

Figure 1b

Ceva's Theorem is usually NOT mentioned in Elementary Geometry courses. This is an unfortunate fact since:

1. the theorem generalizes several statements studied in high school geometry (the corollaries discuss later in this article);
2. its proof is fairly simple, and uses just material traditionally covered in high school geometry (such as similar triangles, ratios, area of a triangle);
3. the statement of the theorem can be investigated using free dynamic geometry software GeoGebra.

Remark: The lines $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, $\overleftrightarrow{CF}$ [Figure 1] are called *Cevian lines* [pronounced: *chev'ian*] of V. The triangle ΔDEF is called the *Cevian triangle* for the original ΔABC.

Proof of Ceva's Theorem: Let ΔABC be a given triangle, and D, E, and F be points that lie on lines $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$, and $\overleftrightarrow{AB}$ respectively. Without loss of generality we assume that the points D, E, and F lie on the sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ of the ΔABC [Figure 1a]. The other case (when two of the points D, E, and F lie outside the sides of the triangle [Figure 1b]) can be proven using similar arguments (note that in this case some of the sensed ratios are negative).

First, let's assume that $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$ intersect at a point V as in Figure 2.

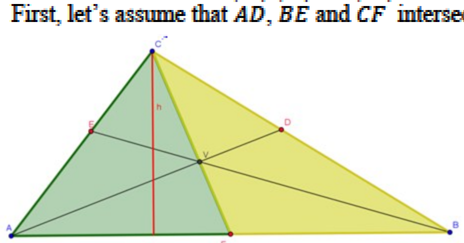

Figure 2

Since the triangles ΔACF and ΔBCF have the same height h [Figure 2], then

$$\frac{area(\Delta ACF)}{area(\Delta BCF)} = \frac{1/2 \cdot h \cdot AF}{1/2 \cdot h \cdot FB} = \frac{AF}{FB}. \qquad (2)$$

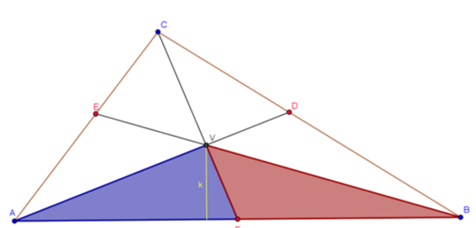

Figure 3

Similarly, Figure 3 shows that ΔAVF and ΔBVF have the same height k, hence

$$\frac{area(\Delta AVF)}{area(\Delta BVF)} = \frac{1/2 \cdot k \cdot AF}{1/2 \cdot k \cdot FB} = \frac{AF}{FB}. \qquad (3)$$

Let's denote $\frac{AF}{FB} = r$. Then, using (2) and (3) we get

$$area(\Delta ACF) = r \cdot area(\Delta BCF), \text{ and} \qquad (4)$$

$$area(\Delta AVF) = r \cdot area(\Delta BVF), \qquad (5)$$

respectively. Subtracting (4) and (5), we obtain

$$area(\Delta ACF) - area(\Delta AVF) = r \cdot \big(area(\Delta BCF) - area(\Delta BVF)\big). \qquad (6)$$

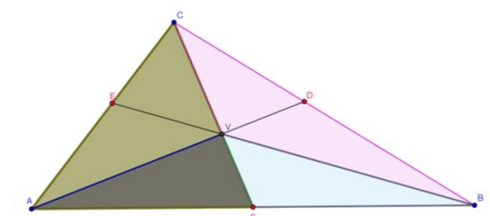

Figure 4

Comparing the areas of the triangles that appear in equation (6) [Figure 4], we notice that the equation (6) can be rewritten as

$$area(\Delta CVA) = r \cdot area(\Delta CVB), \text{ i.e.}$$

$$\frac{area(\Delta CVA)}{area(\Delta CVB)} = r = \frac{AF}{FB}. \qquad (7)$$

Similarly,
$$\frac{area(\Delta AVB)}{area(\Delta AVC)} = \frac{BD}{DC}. \qquad (8)$$
and
$$\frac{area(\Delta BVC)}{area(\Delta BVA)} = \frac{CE}{EA}. \qquad (9)$$

Multiplying the equations (7), (8) and (9), gives

$$\frac{AF}{BF} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = \frac{area(\Delta AVC)}{area(\Delta BVC)} \cdot \frac{area(\Delta AVB)}{area(\Delta AVC)} \cdot \frac{area(\Delta BVC)}{area(\Delta AVB)} = 1,$$

as required. We proved that if the lines $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$ are concurrent (intersect at a point) then equation (1) is satisfied.

Conversely, suppose that the points D, E and F satisfy the equality (1), i.e. $\frac{AF}{BF} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1$. Assume that the lines $\overleftrightarrow{AD}$ and $\overleftrightarrow{BE}$ intersect at a point P, and that the line $\overleftrightarrow{CP}$ intersects $\overline{AB}$ at G. By the first part of the proof, since the lines $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, and $\overleftrightarrow{CG}$ are concurrent they satisfy the equation

$$\frac{AG}{GB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1. \qquad (10)$$