Don Clark—Dr. Hugo Rossi

Dr. Hugo Rossi's prolific career in mathematics spans over five decades. He earned his Ph.D. from Massachusetts Institute of Technology in 1960 and has held academic appointments with the University of California at Berkeley, Princeton University, Brandeis University and the University of Utah, all while serving on a national level with organizations like the American Mathematical Society, Mathematical Association of America, and Journal of Mathematics. People who work with Dr. Rossi (he prefers Hugo) love him; an impressive fact given much of his career has involved bringing people with very various perspectives together to move mathematics and mathematics education forward. Now in his seventh retirement (retirement being the one thing Dr. Rossi does not seem to do well) he is a Professor Emeritus with the U's Math department, Associate Director for the Center for Science and Mathematics Education, PI on a 7th and 8th Grade UCS Math Text, and a member or chair of several U, state and national committees.

Utah Mathematics Teacher
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What is Ethnomathematics?

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Call for Articles

The ***Utah Mathematics Teacher*** seeks articles on issues of interest to mathematics educators, especially K-12 classroom teachers in Utah. All are encouraged to contribute articles and opinions for any section of the journal. Some of the features are: UCTM Leader Spotlight; Letter from the NCTM President; Letter from the UCTM President; Professional Development, Mathematics for English Language Learners; Puzzle Corner; Recommended Readings and Resources; Utah Core State Standards and Implementation; College and University Research; and others.

Teachers are especially encouraged to submit articles including inspirational stories, exemplary lessons, beginning teacher ideas; or managements tools. Sample ideas are (but not limited to) focused on teachers or districts who have successfully implemented the Utah Core, Inquiry based calculus, and new math programs K-12. Manuscripts, including tables and figures, should be typed in Microsoft Word and submitted electronically as an e-mail attachment to Christine Walker (Christine.Walker@uvu.edu). A cover letter containing author's name, address, affiliations, phone, e-mail address and the article's intended audience should be included. Items include, but are not limited to, NCTM affiliated group announcements, advertisements of upcoming professional meetings, and member updates.

George Shell—Melody Apezteguia

Graduating from Brigham Young University in 1993, I earned a degree in Mathematics Education. Furthering my education, I completed my masters' degree at Prairie View A&M University (in Texas) in Curriculum and Instruction with an emphasis in mathematics in 2005. I have been privileged to teach for 21 years in Idaho, Texas, and Utah. Not only do I provide time in the classroom, I look for other opportunities to serve, continually helping on district committees. Within 2 different districts, I have served for the past 12 years on district level committees. I am currently on the Alpine School District Steering Committee for mathematics. In addition, I enjoy teaching professional development courses and trainings both at American Fork High School (AFHS) and at the district level. I have taught courses for the new core, Smartboard use, and TI calculators. Most recently, I served on the lead team for our school's accreditation visit which included the implementation of a data study, professional development training, writing the final report, and organizing materials and supervising the accreditation team's visit.

Striving to be an example and lead wherever I can, I have been a Curriculum Team Leader and department chair for more than 7 years and currently serve as a mentor teacher to the American Fork High School faculty, particularly the Math Department. This year my mentoring role has expanded to being the Lead Curriculum Team Leader at AFHS. This position enables me to work with other Curriculum Team Leaders in my school strengthening our school curriculum through a focus on increasing student engagement, higher order thinking strategies, and implementing and using formative assessment strategies to improve instruction. My dedication to students and education has been recognized through my nomination for the Presidential Award for Excellence in Mathematics and Science Teaching. Last school year, I was selected by Utah's Presidential Scholar in Science as the teacher that most influenced her in high school.

Karl Jones—Joshua Craner

I was born and raised in Salt Lake City as the 3rd of 5 children. My siblings and I were homeschooled during our elementary and junior high years. I then graduated from West High School in 1999. Directly following my high school graduation I attended Westminster College where I earned a Bachelor of Arts Degree with a major in Elementary Education and minors in Special Education and Music. After I taught for a few years I attended the University of Utah where I earned a Master of Education Degree in the area of Teaching and Learning with a focus on Literacy. My entire teaching career has been at Emerson Elementary School in the Salt Lake City School District. I have taught a K-1 Behavior Support Class, a First Grade class, and a 2-4 Behavior Support Class; which I am currently in and loving. I cannot say enough about the amazing professionals I am privileged to teach alongside. They are truly topnotch. During the 2009-2010 school year I took a sabbatical and volunteered for a year at an orphanage/boarding school in a small village in Uganda. It was an amazing experience that changed my life. Some of my passions, outside of teaching, are my family, music and the outdoors.

Muffet Reeves—April Leder

I am the proud mother of 3 daughters and 6 grand children. I have been in education for 19 years teaching both third and fourth grade. I have two master's degrees, one in Curriculum and Instruction and another in Administration, and an Elementary Math Endorsement. I was Alpine School District's Math Specialist for 7 years. I was responsible for the training of 1800 teachers. I am an advocate for all children being taught to understand math concepts, not just use rote procedures and memorize their math facts. Our students will have jobs that require communication skills, problem solving, and flexibility to solve a problem a different way if the first way doesn't work. Math gives us the opportunity to teach using these skills.

Presidential Award Finalists

Karen Feld

I graduated from Utah State University in 2005 and started my first job at Pleasant Grove Junior High. I love teaching junior high students and find that my days are always an adventure. One year after I started teaching at PGJR, I started teaching developmental mathematics classes at UVU. A few years after that I decided to go back to school and get my master's degree, while still teaching at PGJR. I received my degree from Western Governor's University in Math Education. I also teach teacher development classes for the state of Utah.

I absolutely love my job. It is always a challenge. Every day I get to work with students who struggle and help them find clarity. What a pleasure it is to see the look of understanding on a student's face after struggling with a concept. I truly am blessed to do what I do.

Nathan Auck is a math teacher at Horizonte Instruction and Training Center. He earned his Environmental Science degree from The Ohio State University, graduating distinguished and with honors. Shortly after university, Nathan moved west to Utah to pursue outdoor wilderness adventures and inspire youth. His journey has taken him from working with at-risk youth in the wilderness (Second Nature Wilderness Program) to private school (Realms of Inquiry Private School) to public school (Horizonte). Along the way, Nate has expressed his adventurous side by walking 2750 miles from Mexico to Canada on the Pacific Crest Trail, climbing the 3300 foot tall sheer granite cliff of El Capitain and traveling to over 30 international countries in search of perspective, insight and growth. He has had the honor of winning teacher of the year at Horizonte, beginning his National Board Certification and serving a population of students who often struggle to find advocates. He currently resides in Salt Lake City, where he spends his free time working wood with hand tools, gardening, running, climbing, skiing, boating and eating local food with his beloved wife, Sara.

Vicki Lyons loves teaching math to her amazing and wonderful students! She has taught high school mathematics, including Algebra I through AP Calculus and Statistics, at Lone Peak High School in Highland, Utah ever since the school opened in 1997. Prior to that she taught one year at American Fork High School and three years at Ricks Junior College in Idaho. She is National Board Certified and recently she filled a 2-year assignment as a Clinical Faculty Associate teaching at Brigham Young University.

Vicki has designed and facilitated several workshops for the Utah State Office of Education including the Common Core Academy for Secondary Mathematics and she has spoken at many conferences. In the summer she has the delightful opportunity to work as a staff member at the Park City Mathematics Institute Secondary Schools Teachers Program, as a workshop facilitator for the Committee on Teachers as Professionals (c-Tap), and as a Reader for the AP Statistics exam.

Vicki has a MA and BA in Mathematics Education and a BS magna cum laude in mathematics from Brigham Young University. In her extra time, she loves nature and being outdoors, and truly enjoys walking to and from school everyday. She is also a mother and a grandmother and her happiest times are spent with her family that she adores.

Utah Mathematics Teacher

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Problem to Ponder, taken from www.NCTM.org
President's Corner

It's as Easy as Falling Off a Cliff, or Is It?

In a kingdom long ago, a king decided to let chance determine whether persons who committed major crimes would be allowed to live and stay in the kingdom or would fall to their deaths off a steep cliff. Offenders would be placed blindfolded at the edge of the cliff, and then for the rest of their lives, they would proceed to take a step forward, toward the cliff's edge, or a step backward, away from the edge, thus saving themselves—at least for the time being.

A spinner, to be spun by a favorite of the king, would determine whether the offenders stepped forward or backward. On the first spin, a step toward the cliff would send the blindfolded criminal right over the edge. A step away from the cliff would take the offender two steps back from the edge. But then the king's favorite would get to take another spin, randomly determining the offender's next step. And so on...

The king, being a "merciful" ruler, wanted the criminal to have a sporting chance of .5 of not going over the cliff. So he asked the court mathematician, "How should the spinner be divided for stepping toward the cliff and stepping away from the cliff so that an offender has a .5 chance of surviving indefinitely?"

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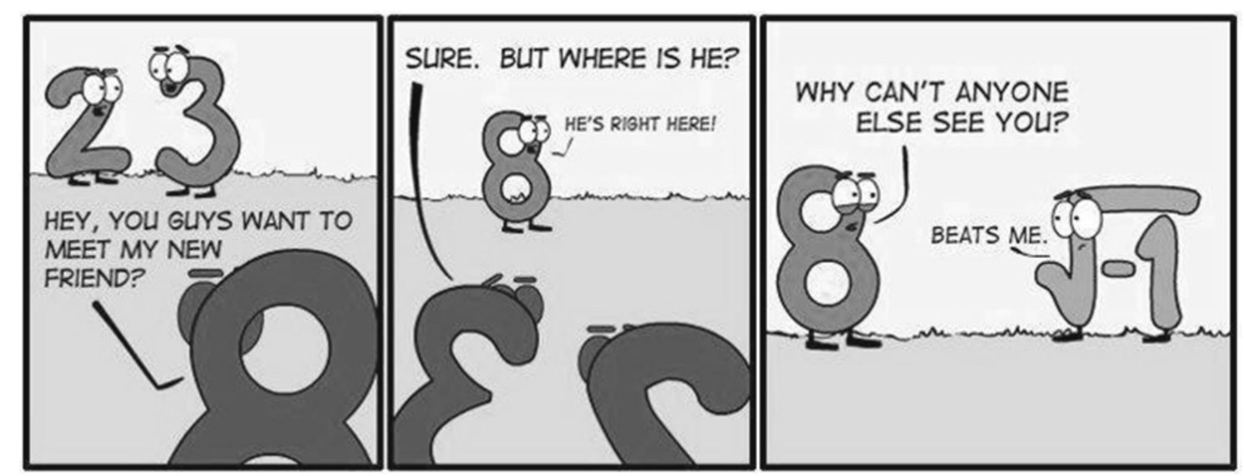

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Presidents Message

Travis Lemon, UCTM President

How are you doing? No seriously, how are you doing in your classroom? What practices and teaching moves are you focused on this year? Are you doing something different than you have done in past years? If not, why not? Are you working to promote meaningful mathematical practices in your classroom? How are you doing this ever increasingly important task?

With the new core standards containing the eight Mathematical Practice standards it is increasingly important for us as mathematics educators to reflect on our teaching practices and to think about how they promote Mathematical Practices for students. Does your instruction provide opportunities for learning that are deep, enduring and relevant? How do you know?

My purpose for asking these questions is twofold: first, to promote self-reflection and a personal assessment of your professional practice. Second, to draw attention to the importance of teaching for understanding and a greater depth of knowledge. More than ever before we need to promote reasoning and sense making. The students we teach are living in a world that moves at an ever-increasing faster pace. They need skills and abilities but they also need well-founded understandings that promote flexibility in thinking and reasoning. More than ever before our students need to know how to think critically, problem-solve and act in accordance to well-developed plans.

I call on all teachers of mathematics to reflect on their teaching practices and on their personal efforts to increase student learning and determine how improvements might be made and then act to make those instructional improvements that will promote greater depth of understanding. Do more than listen and observe, act to implement some of the quality instructional principles and frameworks that researchers have provided and that NCTM has promoted for over two decades. Seek to learn and then to provide meaningful opportunities for your students to learn. Engage in professional learning communities, professional organizations and professional practices that will allow for your instructional techniques to improve and to meet the needs of the learners you serve on a daily basis.

Travis Lemon
President, UCTM
2012—2013

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Making Mathematical Connections

by NCTM President Linda M. Gojak
NCTM *Summing Up*, October 3, 2013

One of the most memorable moments I had in teaching mathematics occurred in a fifth-grade class. We began the year using rectangular arrays as a model to develop the concept of prime and composite numbers. We hung student-made posters of the numbers from 1 to 100 with representations of arrays and lists of factors for each number around the room. By the end of that unit all my students had mastered multiplication facts and could factor with facility as we began our work with fractions. The connections among concepts and the use of concrete representations certainly led to deeper understanding. Later that year, students worked with a variety of models to find area and perimeter of rectangles and extended that experience to find the areas of triangles, parallelograms, and trapezoids. Most students were able to generalize a formula, albeit not always the most efficient, for each polygon. One day, a student commented that this was just like what they had studied at the beginning of the year. When I gave a puzzled look, the class pointed to the posters still on the wall from our first unit of study and said, “You know, that factor and multiple stuff.” I had a new appreciation for the power of providing experiences that enable students to make connections among mathematical ideas. My students remembered and understood the mathematics that we had studied months earlier!

Since that experience I have given much thought to the Process Standards in Principles and Standards for School Mathematics, and their impact on teaching. With the current focus on progressions and trajectories of content standards, the potential of the Connection Standard (NCTM, 2000) continues to pique my interest. It’s a powerful standard:

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Draw Connections and Summarize the Discussion

The first four of the five practices mentioned above (Anticipating, Monitoring, Selecting, and Sequencing) work to set up the discussion, whereas Connecting is primarily meant to occur during the discussion. Rather than having mathematical discussions that consist of separate presentations of different strategies and solutions, the goal is "to have student presentations build on one another to develop powerful mathematical ideas" (Smith & Stein, 2011, p. 11). The teacher supports students in drawing connections between their solutions and other solutions in the lesson. The discussion should come to an end with some kind of summary of the key mathematical ideas. The students ideally leave with "residue" from the lesson, which provides a way of talking about the under- standings that remain when the activity is over (Hiebert et al., 1997).

Concluding Thoughts

In this brief summary, various guidelines and tools were presented to support teachers' efforts to facilitate productive discussions. It is important to recognize that this review only scratches the surface of a growing body of work. Several important areas of this research could not be included here due to space. Some examples include: the teacher's role in classroom discourse (Walshaw & Anthony, 2008); the role of students (Hiebert et al., 1997); the development of mathematical language (see, e.g., Herbel-Eisenmann, 2002; Pimm, 1987); developing lesson goals and planning for productive discussions (Smith & Stein, 2011); using discussion as a formative assessment tool (Lee, 2006); types of questions (e.g., Boaler & Humphreys, 2005) and patterns of questioning (Herbel-Eisenmann & Breyfogle, 2005); equitable participation in classroom discussions (Esmonde, 2009); student motivation to participate in discussions (Jansen, 2006), and so on. There is still much to learn about the conditions under which discussions are productive toward reaching learning goals in mathematics classrooms. The guidelines and tools presented here, however, are intended to provide teachers with a place to begin working on their own goals of facilitating productive and powerful mathematics discussions.

Acknowledgments

The author would like to thank John Pelesko, Tonya Bartell, Beth Herbel-Eisenmann, and the reviewers for their feedback and comments which improved these briefs. The research re- ported in this article was supported with funding from the National Science Foundation ([NSF], Award #0918117, PIs Herbel-Eisenmann, Cirillo, & Steele). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the NSF.

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Instructional programs from prekindergarten through grade 12 should enable all students to—

- recognize and use connections among mathematical ideas;
- understand how mathematical ideas interconnect and build on one another to produce a coherent whole;
- recognize and apply mathematics in contexts outside of mathematics.

Too often, rather than making sense of mathematical ideas, students focus on remembering procedures or tricks. For example, how many students learn "flip and multiply" to divide fractions but have no idea why it works? Often those who understand why the procedure works struggle to apply it in problem situations. The procedure alone often leads to misconceptions. Students who work from rote memory often invert the wrong fraction, forget to change operations, or even apply the rule when multiplying two fractions. The meaning of operations doesn't change from whole numbers to fractions. For example, in the early grades, the understanding that students develop of division of whole numbers often rests on the idea that "9 ÷ 3," for example, asks how many groups of 3 are in 9. As students move to fractions, it is important to provide them with experiences that connect this whole-number understanding to similar examples with fractions: "9/16 ÷ 3/16," for example, asks how many groups of 3/16 are in 9/16. In this way, students gain a deeper understanding rather than depending on a memorized procedure and can apply division of fractions to a variety of problem-solving situations and real-world applications.

Many teachers use manipulative materials to introduce a new concept. Manipulatives themselves, however, do not ensure understanding. We must provide experiences that support students' efforts to make connections between what they are doing with the materials and the mathematical ideas that they represent. This takes time and teacher expertise. Algebra tiles are not an end to teaching basic algebra concepts—when used appropriately, they provide students with opportunities to connect their work to the concepts. And it is these connections that enable students to make sense of the abstract representations.

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Although it is important to think about the connections among concepts within the grade level or courses that we teach, it is also important to reflect on the connections across grade levels. This work involves thoughtful discussions with other colleagues about the way that concepts are taught and the potential linkages among those ideas. Many of us learned mathematics as isolated pieces of information. Taking a mathematical concept and considering how it originates, extends, and connects with other concepts across the grades will help teachers to develop a deeper understanding. It is then that we can plan instruction that ensures that our students regularly make connections to help them make sense of the mathematics they are learning.

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mathematical goals for the discussion will be achieved by making purposeful choices about the order in which students' work is shared (M. Smith & Stein, 2011). Smith and Stein suggested that teachers can also benefit from a set of moves that will help them lead whole-class discussions. Specifically, they focused on a set of "talk moves" that can be used to support students as they share their thinking with one another in respectful and academically productive ways.

Use Teacher Discourse Moves

In Classroom Discussions, Chapin, O'Connor, and Anderson (2003, 2009) introduced five "productive talk moves," which they described as suggested actions that were found to be effective in "making progress toward achieving [their] instructional goal of supporting mathematical thinking and learning" (p. 11). This claim was based on data from their work in Project Challenge, an intervention project initially aimed to provide disadvantaged elementary and middle school students with a reform-based mathematics curriculum that focused on mathematical understanding, with a heavy emphasis on talk and communication about mathematics. A goal of using the talk moves was to increase the amount of high-quality, mathematically productive talk in classrooms.

Building on Chapin et al. (2003), Herbel-Eisenmann, Cirillo, and Steele expanded this earlier work through a five-year project aimed at supporting teachers' facilitation of classroom discourse through the design of a professional development curriculum program. The curriculum supports secondary mathematics teachers in becoming more purposeful about engaging students in mathematical explanations, argumentation, and justification. A modified set of talk moves serves as a centerpiece of the curriculum. This set of Teacher Discourse Moves (TDMs) is a tool that can help facilitate productive and powerful classroom discourse. As part of the curriculum's overarching goals, productive focuses on how discourse practices support students' access to mathematical content. Powerful refers to how classroom discourse supports students' developing identities as knowers and doers of mathematics. There are six TDMs (cf. the five talk moves), which are defined in such a way that highlights what is special about thinking and reasoning in mathematics class as opposed to any other subject area (Herbel-Eisenmann, Steele, & Cirillo, in press). These six moves are:

- Waiting (e.g., Can you put your hands down and give everyone a minute to think?)
- Inviting Student Participation (e.g., Let's hear what kinds of conjectures people wrote.)
- Revoicing (e.g., So what I think I hear you saying is that if there was only one point of intersection, it would have to be at the vertex. Have I got that right?)
- Asking Students to Revoice (e.g., Okay, can some-one else say in their own words what they think Emma just said about the sum of two odd numbers?)
- Probing a Students' Thinking (e.g., Can you say more about how you decided that?)
- Creating Opportunities to Engage with Another's Reasoning (e.g., So what I'd like you to do now is use Nina's strategy to solve this other problem with a twelve-by-twelve grid.)

The six TDMs can be particularly productive and powerful when they are purposefully used in combination with each other (e.g., Asking Students to Revoice after Probing a Students' Thinking). These moves can be used in conjunction with the Five Practices introduced above.

(c) positions the teacher (rather than the students) as arbiters of mathematical truth; (d) minimizes the cognitive engagement on the part of students; (e) communicates to students that there is only one solution path; and (f) represents premature closure of mathematical exploration (p. 103). As an alternative to telling, the authors put forth the strategy of initiating. Initiating includes but is not limited to the following actions:

- Summarizing student work in a manner that inserts new information into the conversation
- Providing information that students need in order to test their ideas or generate a counterexample
- Asking students what they think of a new strategy or idea (perhaps from a "hypothetical" student)
- Presenting a counterexample
- Engaging in Socratic questioning in an effort to introduce a new concept
- Presenting a new representation of the situation (e.g., a graph to accompany a table of values)

These strategies offer alternatives to directly telling students information so that the teacher can productively move the discussion forward. Another strategy involves allowing the students to share their ideas as the basis of the discussion. Sometimes even incorrect strategies are worth exploring.

Explore Incorrect Solutions

Rather than only allowing correct solutions and strategies to surface in discussions, many teachers have taken steps to reduce the stigma attached to being wrong, thus communicating to students that mistakes are part of the learning process (Staples & Colonis, 2007). Some researchers have found that exploring incorrect solutions can serve as a springboard for discussion. This can give a focus to the discussion and engage students in figuring out why an idea does or does not make sense (Bochicchio et al., 2009). This move has several benefits, including: addressing common misconceptions, refining student thinking, prompting metacognition, and engaging students in developing hypotheses (Bochicchio et al., 2009). Staples and Colonis (2007) found that, in collaborative discussions, it was rare for something to explicitly be identified as "wrong." Rather, students' ideas were treated as "works in progress," and the focus of the teacher's guidance was to help the student and the class extend the idea that had been presented and continue to develop a viable solution collaboratively. Purposefully selecting and sequencing the presentation of student ideas can be an effective way to organize a discussion of both incorrect and correct student solutions.

Select and Sequence the Ideas to Be Shared in the Discussion

One of the primary features of a discussion-based classroom is that, instead of doing virtually all of the talking, modeling, and explaining themselves, teachers must encourage and expect students to do so. To do this effectively, teachers need to organize students' participation (National Council of Teachers of Mathematics, 1991). After monitoring the work of students as they explore the task (described above), teachers can select and sequence the ideas to be shared in the discussion (M. Smith & Stein, 2011). Selecting involves deciding which particular students will share their work with the rest of the class to get "particular pieces of the mathematics on the table" (Lampert, 2001, p. 140). Selecting which solutions will be shared by particular students is guided by the mathematical goal for the lesson and by the teacher's assessment of how each contribution will contribute to that goal. Sequencing is deciding on what order the selected students should present their work. Teachers can maximize the chances that their

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Elementary Strand

NCTM, Student Learning Research Brief

The statistical concepts of average, particularly middles or means, are very powerful in statistics, since on the one hand measures of center are often used in a descriptive role to summarize information about a data set. On the other hand, if a data set is a sample that has been appropriately drawn from a parent population, the sample data might be expected to "mirror" the parent population, and thus the mean of that sample provides some information about the (unknown) mean of the entire population from which the sample was drawn. Better yet, collections of samples and their means can furnish a "likely range" within which the actual unknown population mean is probably located. Measures of center thus can play not only a descriptive role but also an inferential role, since we use information from samples to infer information about populations or to make comparisons between populations. How do our students actually tend to think about the concept of "average"?

Mokros and Russell (1995) interviewed students in grades 4, 6, and 8 in "messy data" situations, using contexts like allowance money and food prices that were familiar to students. These students had been taught the procedure for finding the arithmetic average, so they had some familiarity with computing means. However, Mokros and Russell asked students to work backward from a given mean to some possibilities for the data set that could have produced that mean. For example, in one problem students were told that the mean cost of a bag of potato chips was $1.35, and then they were asked to construct a collection of ten bags whose prices had a mean of $1.35. In searching for students' own preferred strategies while they worked on statistical tasks involving averages, Mokros and Russell identified five different conceptions of average among the students they interviewed: average as mode, average as algorithm, average as reasonable, average as midpoint, and average as point of balance.

Some of these conceptions of average prove to be impoverished, whereas others can lead to students' developing higher levels of thinking about data. Students who focus primarily on modes in data sets have difficulty working backward from the mean to construct a data distribution that has that mean, especially if they are not allowed to use the mean value itself as a data value. Mokros and Russell concluded that such modal-thinking students don't see the whole data set as an entity; they can focus only on individual data values. They also found that students who think of average primarily as an *algorithm* aren't able to make connections between their computational procedures and the original context of the data. Students who think of average as reasonable tend to believe that it is an approximation, not something that one can compute. Even some of the students who had more powerful conceptions of average, such as the midpoint or as the point of balance (though Mokros and Russell found the latter conception to be rare among their interviewees), had difficulty re-creating a set of bags of potato chips with an average price of $1.35 if they weren't allowed to use $1.35 as a data point. One conclusion we can draw from

Mokros and Russell's work is that computational facility with average does not guarantee that conceptual understanding or contextual connections about average will follow in our students. We have to create opportunities for our students to connect back to the original context and to interpret what their computations mean in light of the context.

Konold and Pollatsek (2002) and Waton and Moritz (2000a, 2000b) provide us with further evidence for the variety of ways that students think about the meaning of measures of center. Watson and Moritz's findings suggest that students think predominantly of "middle" when they are asked what average means. For example, when asked what it means for a student to be average, or what average means in the context of "the average wage earner can afford to buy the average home," students most frequently referenced "middles," and then "most," with the mean being a distant third. Watson and Moritz offered strong evidence with a large sample of students that there are developmental trajectories for students' understanding of the concept of average. They suggest that students' conceptual development of average starts with idiosyncratic stories, proceeds to everyday colloquial ideas, then to "mosts" and "middles," and finally to the mean as a representative of a data set.

In their work with secondary and postsecondary students, Konold and Pollatsek postulated four conceptual perspectives for the mean: (1) mean as typical value, (2) mean as fair share, (3) mean as a way to reduce data, and (4) mean as a signal amid noise. From a statistical point of view they argue that "signal amid noise" is the most important and most useful way to think about the mean when comparing two or more data sets. In fact, they recommend that the mean should be first introduced to students in the context of comparing two or more data sets. Children's thinking of average as a typical value arises naturally from their experience, as documented by Watson and Moritz and by Mokros and Russell. Thus, "mean as typical" may be a good starting point for teachers to connect to students' own informal knowledge, and "mean as fair share" can provide a conceptual platform for connecting to the algorithm for finding the mean. For example, "leveling" stacks of cubes of varying heights to make all the stacks of equal height (fair shares) allows students to uncover the algorithm for computing the mean as well as to generate alternative algorithms (Foreman and Bennett 1995). However, for Konold and Pollatsek, these two conceptions, mean as typical and mean as fair share, are limited and closely tied to the "data analysis" part of statistics, whereas the latter two conceptions, mean as data reducer and mean as signal amid noise, are connected to the "inference and decision making" part of statistics.

In decision-making from data, the process of data reduction is crucial in order to locate an informative signal amid the noise of the variability in data. Thus statistics often reports the mean of a data set as a "'representative" for an entire data set. Means also furnish a useful signal for making inferences from samples to entire populations and for comparing multiple data sets. Konold and Pollatsek argue that students do not naturally gravitate to using means to compare data sets or to make inferences from samples to populations and that instruction needs to help students grow past their initial informal conceptions of average by concentrating on the "data analysis and decision making" perspective for averages in which the mean is a representative of a data set. They claim that thinking of average as "typical" or as "fair share" does not provide a helpful basis for making group comparisons, whereas the "data reduction" or "signal amid noise" conceptions are more powerful tools in such inferential settings. Konold and Polletsek's work suggests that students' initial informal conceptions of average as "typical value" or a "fair share" may impede their conceptual development unless teachers help them to move toward more conceptually rich notions of average, such as average as "representative" or average as "signal."

type of discourse is much less teacher-directed and predictable because it is "negotiated" and jointly determined by both teachers and students as teachers pick up on, elaborate, and question what students say (Nystrand, 1990, 1991). These kinds of interactions are often characterized by "authentic" questions, which are asked to get information (e.g., "Can you tell us how you decided the answer was 5?"), not to test what students know and do not know. The primary function of a discussion is to construct group knowledge (Bridges, 1987), and questions are the key to fruitful discussions. The research on questioning is vast; therefore only a brief overview is provided below.

Examine and Plan Questions

Examining one's own questions and questioning patterns is an important start when looking more closely at the classroom discourse (see, e.g., Herbel-Eisenmann & Cirillo, 2009). This examination alone, however, has not been shown to do enough to support teachers in facilitating productive discussions that "focus on mathematical meaning and relationships and make links between mathematical ideas and relationships" (M. Smith & Stein, 2011, p. 50). A single, well-formulated question can be sufficient for an hour's discussion (Dillon, 1983). However, many studies have shown that while teachers ask a lot of questions, these questions frequently call for specific factual answers, resulting in lower cognitive thought (Gall, 1984; Perrot, 2002). Some question-types open up discussion, while others are more "closed" (Ainley, 1987). For example, one type of question takes the form of part-sentences "left hovering in mid-air for the student to supply the missing word or phrase" (Ainley, 1987, p. 24). An example of this 'fill-in-the-blank' type of question is: "This polygon has three sides so we call it a …?" This kind of question is closed, both because it relates to matters of established fact and because the teacher has one "right" answer in mind. On the other hand, it creates the illusion of participation and cooperative activity (Ainley, 1987).

Examples of well-formulated questions are: "What is the relationship between the solutions to a quadratic equation and its graph?" or "Why did you solve the quadratic equation to help you graph the parabola?" To answer to these types of questions, students need to provide more than just one-word answers because the answers are complex and require a deeper level of thinking to give complete answers. More open questions are often better for opening discussion and maximizing the chances of individuals to contribute to the discussion, yet such questions tend to be underused (J. Smith, 1986). It can be useful to plan not only tasks but also good questions in advance of the lesson (M. Smith & Stein, 2011), and to consider what questions we can ask to avoid too much "telling."

Be Strategic About "Telling" Information

In a series of papers titled Arbitrary and Necessary, Hewitt (1999, 2001a, 2001b) urged mathematics educators to consider teaching approaches that allow students to discover the necessary (e.g., that the ratio of a circle's circumference to its diameter is a constant number that is approximately 3.14), while only telling students that which is arbitrary (e.g., that this constant ratio of a circle's circumference to its diameter is denoted as pi (p)). This distinction between what to tell versus what to allow students to discover goes against traditional teaching methods where teachers were typically the deliverers of all information, both arbitrary and necessary.

Lobato, Clarke, and Ellis (2005) pointed out several draw-backs to the "teaching as telling" practice. Telling is undesirable when it: (a) minimizes the opportunity to learn about students' ideas and strategies; (b) focuses only on the procedural aspects of mathematics;

were set up to require a high level of cognitive demand tended to decline into less demanding student engagement more than half of the time that they were implemented. Teachers can work to maintain the cognitive demand of a task by investing time before the lesson in the recommendation described next.

Anticipate Strategies That Students Might Use to Solve the Tasks and Monitor Their Work

Teaching in a manner that productively makes use of students' ideas and strategies that are generated by high-level tasks is demanding. It requires knowledge of mathematics content, knowledge of student thinking, knowledge of pedagogical "moves" that a teacher can make to lead discussions, and the ability to rapidly apply all of these in specific circumstances (M. Smith & Stein, 2011). To support teachers in this endeavor, Smith and Stein suggested five practices that are intended to make student-centered instruction more manageable. This is done by moderating the degree of improvisation required from the teacher in the midst of a discussion. Rather than providing an instant fix for mathematics instruction, the five practices provide "a reliable process that teachers can depend on to gradually improve their classroom discussions over time" (Stein, Engle, Smith, & Hughes, 2008, p. 335). The first two of the five practices are anticipating students' solutions to a mathematics task and monitoring students' actual work on the task as they work in pairs or groups.

Anticipating requires considering the different ways the task might be solved. This includes anticipating factors such as how students might mathematically interpret a problem, the array of correct and incorrect strategies students might use to solve it, and how those strategies might relate to the goal of the lesson (M. Smith & Stein, 2011). Anticipating can support teachers' planning by helping them to consider, in advance, how they might respond to the work that students are likely to produce and how they can use those strategies to address the mathematics to be learned.

Monitoring, as described by M. Smith and Stein (2011), is attending to the thinking of students during the actual lesson as they work either individually or collectively on the task. This involves not only listening to students' discussions with their peers, but also observing what they are doing and keeping track of the approaches students are using. Monitoring can support teachers by allowing them to help students get ready for the classroom discussion (e.g., asking students to have an explanation prepared that uses mathematically precise language). It can also help teachers identify strategies that will advance the "collective reflection" (Cobb, Boufi, McClain, & Whitenack, 1997) of the classroom community and prepare for the end-of-class discussion (M. Smith & Stein, 2011). The remaining three of the five practices for orchestrating productive discussions (i.e., selecting, sequencing, and connecting) will be elaborated in later sections of this paper.

Allow Student Thinking to Shape Discussions

In his work on language use in the classroom, Nystrand (1997) argued that people learn not merely by being spoken (or written) to, but also by participating in the discussion about the ideas. This theory of learning is based on the Vygotskian (1978) notion that people learn through social interaction. Discussions can provide students with opportunities to learn by talking with their peers in small groups and by engaging in argumentation, justification, and reasoning in whole-class discussions. In discussion-oriented classrooms, students' responses inform the teacher questions and shape the course of the classroom talk. In particular, the teacher validates particular students' ideas by incorporating their responses into subsequent questions. This

The work of Mokros and Russell, Watson and Moritz, and Konold and Pollatsek presents developmental pathways of students' conceptions of average and suggests conceptual bases to help create teaching and learning trajectories for the concept of the mean. Their work also clearly points out that in fact students do have a rich variety of conceptions of average that we can build on. However, students' conceptions are in transition, and thus they may not understand the important differences in concepts like mean and median, or when the use of a certain measure of center is most appropriate. The teacher plays a critical role in helping students to parse out the best appropriate uses of measures of center (e.g., Zawojewski and Shaughnessy 2000).

We should add one caveat here, lest we become too caught up with the concept of average separate from the rest of statistics. Averages do not exist independent of the distributions of data that they summarize, and in that summary, averages alone can mask a lot of information, namely, the "noise," or variability, in a data set. Centers are only one aspect of a distribution; shape and variability are just as important both in describing data and in aiding in inferential decision making. A close look at the school statistics curriculum, particularly in the United States, reveals that far more time is spent in school on notions of average than on variability or on the shape of data distributions. Often the noise in data, the variability, supplies some essential information that can become lost in a summary statistic like an average (see Shaughnessy and Pfannkuch [2002]). Thus, it is also important for teachers and students to work on describing and analyzing the "noise" in data, to explore variability in data, and not to limit themselves to finding measures of center. We recommend that you also read the companion NCTM research brief to this one that discusses the importance of students' understanding of variability.

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Predications and Probability

Elementary Strand—Investigations
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The "Investigations" department features children's hands-on, minds-on mathematics explorations and presents teachers with open-ended investigations to enhance mathematics instruction. These tasks invoke problem solving and reasoning, require communication skills, and connect various mathematical concepts and principles. The ideas presented here have been tested in classroom settings.

A mathematical investigation is—

- multidimensional in content;
- open ended, with several acceptable solutions;
- an exploration requiring a full class period or longer to complete;
- centered on a theme or event; and
- often embedded in a focus or driving question.

In addition, a mathematical investigation involves processes that include—

- researching outside sources;
- collecting data;
- collaborating with peers; and
- using multiple strategies to reach conclusions.

Although this department presents a scripted sequence and set of directions for a mathematical exploration for the purpose of communicating what happened in this particular classroom, Principles and Standards for School Mathematics (NCTM 2000) encourages teachers and students to explore multiple approaches and representations when engaging in mathematical activities. Each investigation will come alive through students' problem-solving decisions and strategies in the readers' own classrooms. As a result of their exploration, students will incorporate their reasoning and proof skills as they evaluate their strategies. The use of multiple approaches creates the richness that is so engaging in an investigation; it also helps students find new ways of looking at things and understand different ways of thinking about a problem.

The activities in the Predictions and Probability investigation lead students to discover the difference between experimental and theoretical probability and to develop reasonable prediction strategies from a sampling experiment. Students begin by using sample picks to predict the colors of marbles or cubes in a basket. Then they use the Probability Simulation application on the TI-73 graphing calculator (see **fig. 1** for instructions) to model various configurations that are the outcome of choosing different-colored marbles from a bag. (Alternatively, they may use actual marbles or centimeter cubes.) Students have several experiences of choosing with replacement and then using that data to predict the marble

Attend to the Classroom Culture

The Discourse Project was a five-year, professional development–based study aimed at understanding how mathematics teachers' attention to their classroom discourse could impact their beliefs and practice over time (see Herbel-Eisenmann & Cirillo, 2009). An important realization that teachers involved in the project had was that if they wanted to change the classroom culture by moving students toward a more open, student-centered discourse, they needed to invite their students to participate in this shift. For example, in a book chapter focused on her action research in the Discourse Project, middle school teacher Jean Krusi (2009) wrote about how she involved her students by asking them what makes a good classroom discussion. Together, Krusi and her students constructed a list of five norms for classroom discussion: "Everyone is listening; Everyone is involved; Everyone puts out ideas; No one is left out," and "Everyone is understanding—if not at the beginning, then by the end" (p. 121). Krusi found that, in addition to emphasizing these kinds of social norms, she also needed to mention mathematical norms, such as what counts as evidence in mathematics. As the school year came to a close, students commented that they were participating more compared to the beginning of the year, and that they thought that the discussions were fun.

This example from Krusi's class is consistent with other recommendations from the literature. For example, Chapin and O'Connor (2007) insist that the most critical condition that will support both language and mathematics development is for teachers to establish conditions for *respectful discourse*. Similar to Krusi's student-generated norms, Hiebert et al. (1997) proposed the following norms of the classroom culture: Tasks must be accessible to all students; every student must be heard; and every student must contribute. Discussion is most productive when these kinds of prerequisite conditions of respectful and equitable participation are established in advance (Chapin & O'Connor, 2007). As mentioned above, accessible, high level tasks are also a critical element of a good discussion.

Choose High-Level Mathematics Tasks

Stein et al. (2000) defined a mathematical task as a mathematical problem or set of problems that address a related mathematical idea or concept. The nature of mathematics tasks chosen by the teacher is a critical element to facilitating productive discussions for at least two important reasons. First, mathematics instruction is typically organized and orchestrated around instructional tasks. More specifically, delivery of content in mathematics classrooms tends to consist of working on tasks, activities, or problems. Second, the tasks with which students engage are a critical factor in what students learn about mathematics and how they learn it (Stein, Remillard, & Smith, 2007). The relationship between good tasks and good discussions is simple: If we want students to have interesting discussions, we need to give them something interesting to discuss. Activities with a "low floor" (i.e., mathematics knowledge prerequisites are kept to a minimum) and a "high ceiling" (i.e., mathematics activities can be extended to include complex ideas and relationships) tend to create mathematics experiences worth talking about (Gadanidis, Hughes, Scucuglia, & Tolley, 2009) and give more students an entry point into the discussion. Supporting productive discourse can be made easier if teachers work with mathematical tasks that allow for multiple strategies, connect core mathematical ideas, and are of interest to the students (Franke, Kazemi, & Battey, 2007).

Past research has shown that teachers can find it difficult to maintain the cognitive demand of high level tasks. For example, in their study, Stein et al. (1996) found that tasks that

What Are Some Strategies For Facilitating Productive Classroom Discussions?

General Strand
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NCTM Discussion Research Brief, 2013

One area that has been given a great deal of attention in the mathematics education literature, particularly over the past 25 years, is classroom discourse. This is evident not only in the body of published articles but also in the many policy documents calling for more student talk in mathematics classrooms (see, e.g., NCTM's *Principles and Standards for School Mathematics* [NCTM, 2000] and the Common Core State Standards [NGA Center and CCSSO, 2010]). Although these documents often use different language to describe their communication standards, they are all based on the common assumption that students learn mathematics best when they are given opportunities to speak about mathematics using the language of mathematics. Discussion, which is promoted in all of the documents, can therefore provide students with opportunities to communicate mathematically.

Because many of us learned to teach through the "apprenticeship of observation" (Lortie, 1975) in traditional class- rooms, calls to shift from recitation to discussion -based lessons can be challenging. Many teachers are understandably unsure and overwhelmed by the call to use rich tasks and to facilitate discussions in mathematics class (see, e.g., Ball, 1993; Chazan, 1993). Over the past 15 years, fortunately, the field has begun to tackle the problem of providing teachers with guidelines and tools to support the facilitation of productive classroom discussions. Nine strategies for facilitating productive discussions are listed below and are discussed in more detail throughout the remainder of the paper.

- Attend to the classroom culture
- Choose high-level mathematics tasks
- Anticipate strategies that students might use to solve the tasks and monitor their work
- Allow student thinking to shape discussions
- Examine and plan questions
- Be strategic about "telling" new information
- Explore incorrect solutions
- Select and sequence the ideas to be shared in the discussion
- Use Teacher Discourse Moves to move the mathematics forward
- Draw connections and summarize the discussion

distribution in the bag, leading to a discussion of definitions for both experimental and theoretical probability (see **fig. 2**). After identifying probability situations as either experimental or theoretical, the investigation expands to predicting probabilities from data in circle graphs. This investigation addresses NCTM's Content Standard for Data Analysis and Probability (NCTM 2000) and culminates with a discussion of strategies for making reasonable predictions of theoretical probability.

The Investigation

Learning goals, rationale, and pedagogical context

Understanding probability and using it appropriately are necessary skills for functioning as an informed citizen. Principles and Standards outlines the importance of reasoning with probability: "Instructional programs from prekindergarten through grade 12 should enable all students to understand and apply basic concepts of probability" (NCTM 2000, p. 400). Probability can be determined theoretically or experimentally. This investigation allows students to explore both types of probability through marble- picking experiments and to "encounter the idea that although they cannot determine an individual out- come … they can predict the frequency of various outcomes" (p. 181). Computer or calculator simulations may help students confront misconceptions or inaccurate intuitions about probability (NCTM 2000; Tarr, Lee, and Rider 2006). This investigation uses the Probability Simulation application on the TI-73 graphing calculator. The application is also available for the TI-83 Plus and TI-84 Plus families of graphing calculators. Free downloads are avail- able at www.education.ti.com. Alternatively, hands- on materials can be used. In the interest of time, those using hands-on materials may want to decrease both the number of marbles in the containers and the number of marbles picked in the various activities. The lessons were field tested with two groups of students in the Buffalo Public Schools: a sixth- grade class at the Mathematics, Science, Technology School and a group of Community School no. 53 sixth-graders attending the High School Ahead Mathematics Academy, a Saturday morning tutoring program.

Objectives of the investigation

Students will—

- differentiate between experimental and theoretical probability;
- understand why experimental probabilities do not
- always match the theoretical probabilities; and,
- use a variety of strategies to make reasonable predictions of a sample space based on experimental data.

Materials

For all three lessons, each student needs a TI-73 graphing calculator with the Probability Simulation application loaded on it. The teacher needs an overhead TI-73 graphing calculator unit. Alternatively, instead of a calculator, each student could use a container and a total of thirty objects such as marbles or centimeter cubes representing three different colors. Students will work in cooperative groups of three or four in Lessons 1 and 3 and with a partner in Lesson 2.

Lesson 1
For each student—

- one copy of **activity sheet 1**, "Mystery Marbles"

For each group—

- 3 paper lunch bags
- 90 centimeter cubes (30 blue, 30 green, 30 red)

Lesson 2
For each student—

- three or four copies of **activity sheet 2**, "Find My Marbles"
- if not using graphing calculators:
  - ¤ 1 paper lunch bag
  - ¤ 90 centimeter cubes (30 each of 3 different colors)

Lesson 3
For each student—

- one set of probability sort cards (see **fig. 3**)
- one circle graph handout (see **fig. 4**)
- one recording sheet (see **fig. 5**)
- if not using graphing calculators:
  - ¤ 1 paper lunch bag with 20 centimeter cubes distributed among 5 different colors (for the teacher only)

Previous knowledge

The students who did this investigation were familiar with calculating simple probability and with the concept of a sample space. They also had experience making and justifying predictions.

Lesson 1: Mystery Bags

In the first session, students predict—for each of three different bags—the number of cubes of each color in the bag. Before the class starts, assemble the following three bags of cubes for each group of students (do not let the students see the cubes):

- Bag One—2 blue, 2 green, 6 red cubes;
- Bag Two—1 blue, 6 green, 3 red cubes; and
- Bag Three—7 blue, 2 green, 1 red cube(s).

Appendix A
An Exchange with Dr. D'Ambrosio

Figures 1 and 2 document an exchange between the author and Dr. Ubiratan D'Ambrosio, an early contributor to the study of ethnomathematics, at UNIBAN, São Paulo, Brazil, 4 October, 2011.

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In Scenario 1, ask students to make a prediction without any data (i.e., without drawing any cubes from the container): "Bag One contains ten cubes. Some are blue; some are green; some are red. How many cubes of each color are in the bag? Make a prediction without looking or reaching into the bag. Explain how you got your answer."

Without additional information, many students assumed the number of each color must be roughly the same. Although the actual assignment of colors differed, exactly half of the students predicted a distribution of 4–3–3 and another 42 percent predicted 4–4–2. Only 8 percent predicted a different distribution. Damon's response indicates how strongly some students hold the misconception of the colors being equally distributed. He wrote, "3 red, 3 blue, 3 green; 1 may be of any color." As expected, most of the students explained that they just guessed. Lindsey provided a detailed rationale for the 4–3–3 answer: "Because if you divide ten by three, you get 3.33 … so I took out the decimal." She then indicated she could add the remaining one to any of the three colors. The students were surprised to find out there were two blue cubes, two green cubes, and six red cubes; many students commented that they had not made good guesses. Explorations with Bag Two and Bag Three will counteract this mindset.

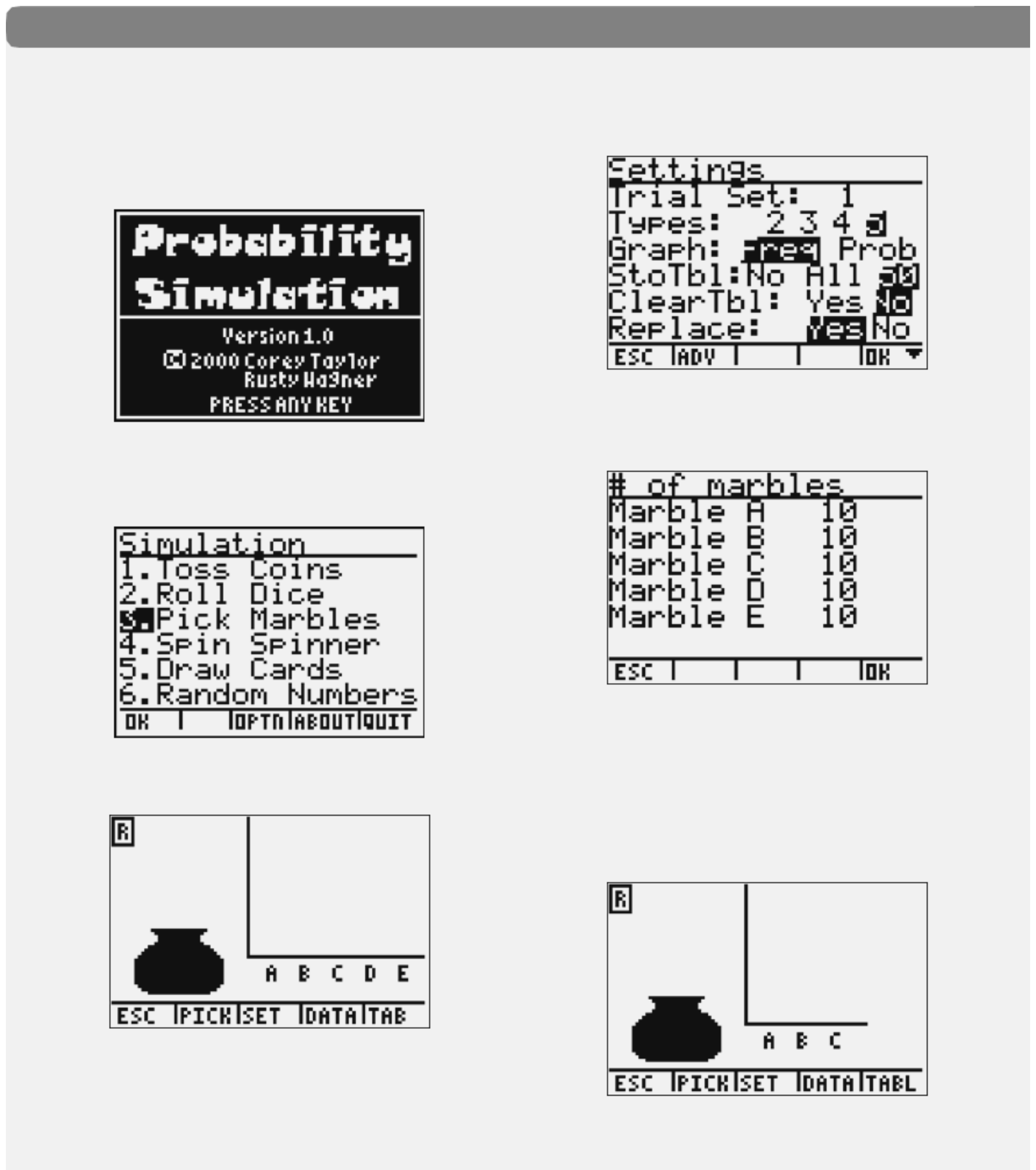
In Scenario 2, ask the students to make a pre- diction, but reveal all of the cubes first (they need not predict, only report, the color distribution): "Bag Two contains ten cubes. Some are blue; some are green; some are red. Take turns picking a cube and displaying it on your desk. After all ten cubes are on the desk, tell me how many of each color are in the bag and explain how you got your answer."

Ask the students to discuss how they knew the answer and how this scenario differs from the first one. It is important that students understand the difference between predicting and giving an answer based on a complete set of data.

In Scenario 3, ask students to pick cubes with replacement for a total of ten picks (i.e., the students have experimental data to use as they predict the contents of the bag): "Bag Three contains ten cubes. Some are blue; some are green; some are red. Take turns picking a cube, recording its color, and then putting it back in the bag. Repeat this ten times. You may want to use tally marks to record the picks. After ten picks, tell me how many cubes of each color are in the bag and explain how you got your answer."

In one class, picking with replacement from Bag Three (seven blue cubes, two green, and one red) produced five blue cubes, four green, and one red. Over half of the students predicted the bag contained five blue cubes, four green, and one red, the exact breakdown of the cubes that were selected. Most of the remaining students made an adjustment of plus or minus one, predicting six blue cubes, three green, and one red; or four blue cubes, four green, and two red. In the ensuing discussion, most students either had no explanation for their prediction or indicated that they guessed. Only a few connected the picks to their predictions by using explanations such as Nelson's: "Because she [the teacher] picked more blue [cubes] than any- thing" or Dyneal's: "Because we kept on getting blues and greens and only got one red." Even fewer mentioned the fact that the same cube could have been pulled out more than once. You may want to reinforce the concept that just because the number of picks equals the number of objects in the container, the results from sampling will not necessarily be identical to the contents of the container.

Give each student a copy of **activity sheet 1**. After reading the description of the bag of marbles (ten each of three different colors for a total of thirty marbles), the students should record their individual predictions in the table, explain their thinking, and then discuss their responses with their group members. Then each student should use the Probability Simulation application on the TI-73 graphing calculator to set up a bag of thirty marbles with ten each of three different colors for picking with replacement (see **fig. 1** for instructions). Have each student use the calculator simulation to pick thirty marbles, record the results in the table, and answer the remaining questions on the sheet. The right arrow key will display the number of marbles for each bar in the graph. If your students are not using the calculators, have each group take the cubes from the three bags and redistribute them so that each bag contains cubes of all three colors.



This lesson ends with a whole-class discussion of students' conclusions. In one class, just over one-third of the students initially predicted that ten of each color would be picked.

reasons that their predominately African-American school was surrounded by liquor stores (i.e., a wet area), while schools in White, upper-middle class communities were located in dry areas. The students exposed this inequity using mathematics, literacy, and political skills as they produced reports, editorials, charts, graphs, and maps. In this example, the teacher problematized the students' circumstances and provided opportunities for them to use their knowledge and skills to transform the world around them.

Traditionally, an aim of mathematics education has been to prepare students to succeed in the currently existing world. However, mathematics education researchers (Martin, 2003; Secada, 1989) argue that mathematics education needs to empower students to use mathematical knowledge to confront issues of social justice and unequal power relationships. In addition, the level and nature of mathematics required for success today differs significantly from that required in the past, and the requirements will continue to change and advance as technology and the needs of society evolve. For students to assume a productive and successful role in the future, they need to have developed critical thinking skills that will enable them to effect change in positive ways. Therefore, in addition to providing instruction necessary for success in the current system, mathematics educators need to provide opportunities to use mathematics to confront obstacles to their success (Martin, 2003; Secada, 1989).

Final Thoughts

Mathematics educators' shift towards viewing culture as playing a major role in the learning of mathematics represents a new way to think about social issues in the schools. Historically, scholarly mathematics was made available only to those of elite status. To this day, mathematical competence acts as a gatekeeper to higher-paying jobs and social classes. In response to notions that scholarly mathematics is superior to everyday mathematics, a handful of ethnomathematics researchers have recognized the value of different cultures' mathematical practices and have made attempts to valorize those differences. An examination of differences in mathematical valorization reveals how they are incorporated into issues of power and dominance, especially in the school setting.

In 2000, the National Council of Teachers of Mathematics issued a call for educators to provide more equitable opportunities for students to learn mathematics. They stressed that equity does not mean providing the same experiences for all students, but rather, accommodating differences and providing support to help all students learn mathematics. In light of the cultural implications discussed here, this requires educators to expand their beliefs regarding mathematical competence and to incorporate culturally relevant pedagogy into their teaching practices. Clearly, culture significantly impacts the types of mathematics students learn and the methods used to learn them. As more research is conducted on the nature of this relationship and ethnomathematics, mathematics educators will have more tools to effectively teach diverse populations of students.

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these implications. Culturally relevant pedagogy aims to "produce students who can achieve academically, produce students who demonstrate cultural competence, and develop students who can both understand and critique the existing social order" (1995, p. 474). Below, each of these goals will be discussed in relation to the teaching of mathematics and a mathematics classroom environment.

Students who can achieve academically. An overarching purpose of the educational system is to increase students' knowledge and skill set so that they can function and contribute to society. Standardized tests have become highly regarded by society as a means of assessing this set of knowledge and skills and consequently, allocating intellectual power and privilege. Yet, these tests commonly assess student achievement. Ladson-Billings admits, "No matter how good a fit develops between home and school culture, students must achieve. No theory of pedagogy can escape this reality" (1995, p. 475). However, as discussed previously, standardized tests need not be the sole or primary indicator of success in a mathematics classroom. Mathematics teachers can also gain valuable information as they listen to students' conversations about their solution strategies, examine their written justifications, and observe as students pose, compute, and solve complex problems. In these ways, teachers can be assured of their students' skills to think mathematically, and the students can develop confidence in their own abilities to succeed academically with mathematics regardless of their score on a standardized test imposed by society.

Students who demonstrate cultural competence. Students develop cultural competence as they learn ways to "maintain their cultural integrity while succeeding academically" (Ladson-Billings, 1995, p. 476). Teachers assist students in this development of this second goal of culturally relevant pedagogy when they promote positive interactions within the classroom community. These interactions should value and preserve the dignity of cultural differences, while celebrating the similarities among individuals and communities. Ladson-Billings (2001) also suggests that cultural competence develops in classrooms where the teacher "uses culture as a basis for learning" (p. 98). This means that teachers use students' funds-of-knowledge and prior experience as a pathway to new knowledge and skills. Students need not abandon their cultural values and heritage for the sake of succeeding academically.

As students and teachers acknowledge that everyone has cultural history and assets that can enrich learning in the classroom, a sense of community develops. The teaching of mathematics in such a setting allows for students to discuss their mathematical thinking in a supportive environment. If students believe that their ideas will not be valued, they will be less inclined to contribute to class discussions. However, students become more engaged and successful when they feel that the mathematical content relates to their lived experience and has meaning in their everyday lives. This engagement occurs when students have a well-defined sense of cultural competence.

Students who understand and critique the existing social order. A third goal of culturally relevant pedagogy involves students' ability to critique social inequalities and effectively make change from a cultural perspective. Students' opportunity for growth in this area presumes that the teacher first recognizes negative social structures and the need for change. Mathematics teachers can provide students with the tools to challenge social inequalities when they make knowledge problematic, challenge students' views, and encourage students to think through, justify, and defend their perspectives. Ladson-Billings (1995) describes a group of middle-school students in Dallas who, at the encouragement of their teacher, investigated the

Davonte reasoned, "Because it is thirty marbles, and 30 4 3 = 10," and Robert reasoned, "Because there are ten of each color." Another third of the students made predictions close to 10–10–10, such as 8–12–10 or 9–10–11, explaining that they "used probability" to choose numbers close to ten because there were actually ten of each color. Students whose experimental data varied considerably from ten blue, ten green, and ten red were surprised at first. But most of them reassessed their thinking in light of other students' data from the experiment and the class discussion that followed. During the summary discussion, students should indicate an awareness that when sampling with replacement, the same object may be picked multiple times and that some objects may not be picked at all.

Lesson 2: Find My Marbles

Figure 2 **Kendall's work**

Marble Color	Number of Marbles Picked	Predicted Number in the Bag	Actual Number in the Bag
A	10	1	1
B	63	6	5
C	227	23	24
Total	300	30	30

In the second part of this investigation, students predict the marble distribution in their partner's bag of marbles after drawing marbles with replacement. Before starting, you may want to have students reflect on what they previously discovered about picking with replacement and the actual number of marbles in the bag.

Have students create their own bag of marbles according to the guidelines on **activity sheet 2.** Each bag should contain thirty marbles, divided among three colors, and each student may use the color distribution of his or her choice.

Next, students switch bags or calculators with their partners, who then pick thirty marbles (with replacement) and record the picks in the second column of the chart on **activity sheet 2, Part 1**.

Students should use their data to predict and record the number of each color of marble in their partner's bag and then explain their reasoning.

Finally, they should check the number of each color that were actually in the bag by looking into the bag or by viewing the settings in the calculator application (see steps 3–5 in **fig. 1**). They should also record the actual numbers and their reactions. Students who finish early should change the number of each color in the bag and repeat the activity. Alternatively, you may want to have the entire class repeat the activity. The students should use a new copy of **activity sheet 2** each time they repeat the activity.

Follow this activity with a whole-class discussion of the students' estimation strategies and their answers to the **activity sheet 2** questions. Generally, the students made predictions that were close to, but for the most part not exactly equal to, the numbers of marbles they had picked. Only a few of their explanations indicated any strategy other than guessing based on their picks. However, some students indicated that, based on their picks, they were confident about which color had either the most or fewest marbles in the bag, although they could not tell the exact number. For example, Dominque wrote, "C had the highest bar [on the graph] by a lot, so I guessed twenty-five out of thirty." Similarly, Leatrice picked 5–12–13 and then reasoned, "I think my partner has [fewer] As than Bs or Cs in his bag." The discussion made it clear that the students needed additional guidance to move toward applying number sense to experimental data in order to reasonably predict a sample space. So, the students changed the color distribution of the thirty marbles and repeated the activity once more. This time, they were told that although each bag still contained thirty marbles (and **activity sheet 2** tells them to pick thirty marbles), they were not to pick thirty marbles. The hope was that requiring a number of picks that did not equal the number of objects would move them toward the use of proportional reasoning. Many students picked close to thirty times, for example, twenty-five, twenty-nine, or thirty-five times. Unfortunately, those numbers of picks did not easily lend themselves to proportional reasoning. But some chose numbers such as ten or sixty that they thought would be easy either to multiply or divide to reach thirty. A couple of students chose one hundred picks in order to use percentages in their predictions. Kendall made three hundred picks "because it was easy [to compute]." He then divided his experimental data by ten and rounded to the nearest whole number. (His work is shown in **fig. 2**.) One advantage of using technology is the ease and speed of generating such a large number of picks.

To summarize this part of the lesson, have the class repeat the activity a final time while you are the only person picking marbles. This will ensure that the entire class has a common set of data to use as a reference point for the summary discussion. Generate the picks with the overhead calculator from a bag of marbles that you previously set up on your calculator.

Ask the class to reflect on their experiences with this activity and to generate a definition for experimental probability and one for theoretical probability. This class agreed that theoretical probability was "probability found by using the rules." They described experimental probability as "when you do it and see what happens" and as "hands- on." After agreeing on your definitions, have the students complete **Part 2** on **activity sheet 2** using the class data from the last repetition of the activity. Conclude the lesson by having a whole-class discussion in which the students compare the experimental and theoretical probabilities and the use of each to predict or identify sample spaces.

Lesson 3: Candy Confusion

In the final section of this investigation, students continue to work with experimental and theoretical probability. Ask each student to sort the Probability Sort Cards (see **fig. 3**) into two groups: situations representing experimental probability and those representing theoretical probability. Have them compare and discuss their results with the others in their group.

developed bodies of knowledge and skills essential for household or individual functioning and well-being" (p. 133). Therefore, students' funds-of-knowledge become a major intellectual resource in the classroom. They are in no way inferior to traditional academic expectations. Moll and his colleagues adamantly reject the notion that lack of success in school is attributed to the students, themselves, or to their cultures and communities. Instead, success in school depends the extent to which the teachers support students by connecting their home and community experiences to the schools' academic expectations.

How can teachers bridge in-school and out-of-school mathematics? A challenge for most teachers exists in how to incorporate students' out-of-school mathematical experiences with classroom instructional activities. This task becomes even more challenging when the classroom teacher's cultural background differs significantly from that of the students. Nevertheless, the teacher remains as a pivotal element of students' school experience. In order to provide ample opportunity to learn, teachers need to acknowledge that different discourse patterns may exist between home/community and school, and that these differences do not connote inferior or superior forms of knowledge. A large part of the teacher's role in the classroom, especially in mathematics, is to help students connect scholarly mathematics with everyday mathematics. Many times teachers and students do not recognize the mathematics in everyday activities. By using these everyday activities to contextualize and mathematics problems, teachers can make mathematical concepts more accessible to more students.

Critics may argue that only focusing on everyday or practical mathematics restricts the development of more advanced mathematical concepts. To this critique, Moschkovich (2002) suggests a balance between everyday mathematics and academic mathematics. This balance serves to motivate students by helping them to recognize the value of mathematics as connected to their lived experience. It also helps students to develop the skills necessary to succeed should they choose to pursue more advanced studies in a mathematics-related field.

How should mathematics be assessed? Alternative views of what constitutes legitimate mathematics and how it is addressed in the classroom also necessitate new ways to assess students' mathematical understanding. Traditional means of assessment have focused on what students *should* know and to what extent they are *deficient* in and given concept. This culturally biased approach differs significantly from more culturally inclusive assessments that focus on what students *are* doing. Alternative forms of assessment accommodate students' diverse ways of knowing and their many pathways to understanding mathematical concepts. For example, instead of (or in addition to) a traditional paper-and-pencil exam, students may be presented with a real-life problem and asked to describe their solution strategy in writing or orally. Alternatively, students could demonstrate competence by successfully completing a mathematical task outside of the school setting. Naturally, these assessments should build upon students' experience both inside and outside of school.

Culturally Relevant Pedagogy

Classroom learning environments have significant implications on students' experiences in any classroom, and particularly in mathematics classrooms. The ways in which a teacher structures lessons, assigns tasks, and responds to students' comments send messages, for better or worse, regarding what types of mathematical concepts are valued and how students are expected to interact with one another. Although not specifically focused on mathematics classrooms, the work of Ladson-Billings (1994, 1995, 2001) on culturally relevant pedagogy aligns well with ethnomathematics and provides a theoretical framework through which to consider included or excluded in what the group counts as knowledge. (p. 442)

Often members of different cultural groups think about mathematics in ways that are very different from what is generally accepted as scholarly mathematics (i.e., the *real* way to do mathematics). It is important to note that these different ways of thinking are just as legitimate or sophisticated. For example, in the United States, particular subtraction and multiplication algorithms are taught and valorized as the *real* way to perform these calculations. However, in many parts of the world outside of the United States, different methods for subtraction and multiplication are used. These algorithms are no less mathematically valid, yet they are not given the same value as the *real* way to subtract or multiply. In this way, the dominant White culture in the United States has defined "normal" mathematics and rejected other cultures' methods as inferior. As such, advanced mathematics has become a white male-dominated field.

Ethnomathematics presents implications for classroom teachers in that they must reexamine their beliefs about what counts as legitimate mathematics, how mathematical concepts are taught, and how to assess children's knowledge of mathematics. With these constructs in mind, teachers should also be aware of diverse cultures represented in their classrooms and incorporate culturally responsive practices into their instruction. The next sections will discuss issues and implications that teachers face and how they may address them.

Expanding Mathematical Competence

As classroom teachers consider ethnomathematics and the influence of students' diverse cultural backgrounds on learning, they must acknowledge the classroom as a mediating space between academics and everyday experience. This acknowledgement demands a reexamination of the teachers' previously held beliefs regarding mathematical knowledge, how it is taught, and how it is assessed.

What counts as legitimate mathematics? The previously discussed differences between scholarly and everyday mathematics present classroom teachers with decisions of what types of mathematics to legitimize, promote, or valorize in their instruction. Traditional notions of mathematical competence have focused solely on students' ability to accurately identify basic facts and perform calculations using prescribed algorithms (i.e., the students' ability to achieve a narrowly predetermined standard). Mathematical competence from a cultural perspective, however, expands competence as achievement to competence as "being co-constructed by teachers and students in relation to classroom opportunities to learn and to what students are held accountable" (Diversity in Mathematics Education Center for Learning and Teaching, 2007, p. 413). This co-construction may include alternative ways of thinking about mathematics based on students' experiences inside and outside the classroom. For example, based on his experiences with making purchases in real-life situations, a student may already have mathematically legitimate methods for estimating and adding or subtracting two- or three-digit numbers even though they may be different from the traditional algorithms taught in school. A narrow view of mathematical knowledge would likely exclude these alternative methods from legitimate mathematics. However, teachers with a cultural perspective of mathematical knowledge would find value in them and help students to make connections between different elements of mathematical thinking. Therefore, the opportunities presented for students' learning play a significant role in developing mathematical competence.

Additionally, through their work with students from working-class Mexican communities, Moll and his colleagues (1992) have developed a conceptual framework known as *funds-of-knowledge*. They define funds-of-knowledge as "historically accumulated and culturally

To expand the investigation to another representation, give students a circle graph (see **fig. 4**) representing forty total candy bars of five different types. Ask them to individually estimate the theoretical probability of picking each type of candy bar and then to share their strategies and estimated probabilities with their groups. In one class, two students ignored the graph entirely and initially answered one-fifth for each of the five probabilities, maintaining their initial misconception from Lesson 1. However, the rest of the students used visual estimation strategies to create reasonable estimations for the probabilities, with some even drawing in an additional horizontal line segment through the KitKat section in order to "see where the other fourth is."

Next, ask students how they could find an experimental probability for picking each type of candy bar. As a class, these students decided to use a paper clip held at the center of the circle graph as a spinner, spin forty times, and then use the number of spins in a section divided by forty as their estimate for that section.

The culminating activity for this investigation is for students to use experimental probability to predict how many of each type of candy bar is in a bag containing twenty actual candy bars. Before the start of class, assemble a bag of twenty marbles in five colors to model a candy distribution of your choice. You may use marbles or use the Probability Simulation application set for five colors. Inform the students that the distribution in your bag does not match the circle graph in **figure 4**, so you may want them to put away the circle graphs. Give each student a recording chart (see **fig. 5**) to organize their work. Let each student individually decide how many marbles they would like picked from the bag before making their candy bar predictions, with a maximum of five hundred picks. If you are not using a calculator, you will want to choose a smaller maximum number of picks. Display the actual cubes or the calculator screen on the over- head as you make the picks.

About half the students avoided the use of proportional reasoning by making their predictions after twenty marbles were picked, because "there are twenty in the bag." About 25 percent used fifteen marbles to predict, indicating that "fifteen is close to twenty." The remaining students (about 25 percent of the class) picked numbers slightly larger than twenty: either twenty-five, thirty, thirty-five, or forty-five. The students enjoyed watching the marble simulation and wanted to keep going until five hundred marbles had been picked, even though they had all made their predictions based on sixty or fewer picks. I completed the five hundred picks and displayed the results, but there was no evidence that they used the larger number of picks to modify their predictions.

Figure 4 A sample circle graph

Beyond The Lesson

Students enjoyed using the marble simulation to predict and continued to use it in subsequent classes as time allowed. They changed the difficulty by modifying the number of colors, the number of marbles in the bag, or the number of marbles picked. Also, gathering such experimental data is one method to determine whether or not a game is fair. After a lesson on fair games, some students created their own games based on picking marbles and then used the simulation to verify that the games were fair. Some of them also solved a fair game challenge, Challenge 26, retrieved from www.figurethis.org.

Reflections

Leading this investigation confirmed to me that some probability concepts are not intuitive to students. For example, in Lesson 3, the students saw no value in picking a large number of marbles before making a prediction, although I offered to pick as many as five hundred marbles using the Probability Simulation application. One advantage of using a calculator simulation is its quick generation of large samples of data. These investigations would have taken significantly longer if the students had picked the same number of marbles by hand for each activity. In addition, the students did not have to tally the picks, because the simulation would display the total number of each color picked at any point.

Probability plays a role in the everyday decisions people make. Students develop their own informal concepts of experimental and theoretical probability outside of the classroom. This investigation focused on a series of activities that emphasized gathering experimental data and then making reasonable predictions. The activities and discussion involved in this investigation provide opportunities to confront some common but inaccurate ideas about probability and to progress toward a deeper understanding of probabilistic concepts. Whether students use available technology or hands-on materials, "it is useful for students to make predictions and then compare the predictions with actual outcomes" (NCTM 2000, p. 254) to help correct misconceptions about probability.

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Figure 5 Recording Sheet

You may choose between 1 and 500 picks. How many picks do you want before
You make your predication?
Number of picks?

Type of Candy	Number Picked	Experimental probability	Predicted Number	Theoretical probability
A (Kit Kat)				
B (Milk Chocolate)				
C (Mr. Goodbar)				
D (Peanut Butter Cup)				
E (Special Dark)				
Total			20	

in an artificial school setting. In light of this research, mathematics in the schools should be introduced in contests that allow students to connect with everyday human activities—to experience mathematics in their own world. These implications for the classroom will be discussed in a later section.

Similarly, research among members of the Shoshoni tribe (traditionally inhabiting parts of Utah, Idaho, and Wyoming) in the western United States, reveals a different view of mathematics usage than is taught and valorized in the schools. The study, conducted by Barta et al. (2001), described culturally specific uses of mathematics in traditional living practices among the Shoshoni. Through a series of interviews, tribal leaders discussed their knowledge of the Shoshoni history, culture, customs, and practices. Although the Shoshoni language does not have a word equivalent to the *English* mathematics, the actions of mathematics seem to be embedded into everyday living and linked to the Shoshoni way of tribal life.

Interview questions specifically addressed six practices of mathematics identified as universal by Bishop (1988): *counting*, locating, measuring, designing, playing, and explaining. The counting system was based on groups of ten and ranged from zero to countless or infinity (i.e., more of those than there are hairs on a horse). Activities of counting or mathematical operations always occurred in the context of quantifying objects, people, or tribal events. Interestingly, division (e.g., a hunter bringing a deer to the camp to be divided amongst the families) did not always denote equal parts. Rather, each portion was determined by each individual's need. Even though the portions may not be equal, each person received their share. The *location* of people or objects was recorded using topography and star positions. Most *measurement* devices included body parts, sticks, poles, or strips of rawhide. These individualized measures allowed for proportional appropriateness for different users (e.g., clothes, bow and arrows). To outline a circle for a sweat lodge, one would step off the distance of the radius, and then use a strip of rawhide nailed down at one end to draw the circle. Other measurements such as volume, time, distance, and weight were determined by comparisons to everyday occurrences (e.g., a particular bowl or rock, or the number of "suns"). *Designing* took the form of intricate beadwork or functionality of tools and buildings. As in other cultures, the Shoshoni *played* many games. Almost all of these games incorporated some sort of strategy and keeping track of scores. *Explaining* was a critical part of the Shoshoni tradition because it was the method by which customs were communicated and passed down from generation to generation. The tribal leaders demonstrated this communication as they related the aforementioned mathematical activities.

It is clear from the examples of the Brazilian children and the Shoshoni tribe that mathematical symbolism and representation is peculiar to the culture from which it is derived. When viewed from a different cultural lens those representations may or may not have the same mathematical meaning (Barta et al., 2001). For this reason one culture's way of operating with mathematics may receive a lesser amount of value than that of other culture—especially if one cultural group receives less privilege than the other. These varying degrees of valorization contribute to what Presmeg (2007) describes as "symbolic power" and "symbolic violence" ((p. 442). Symbolic power has traditionally been granted to those groups with cultural, symbolic, and linguistic capital—those exclusively espousing scholarly mathematics. Presmeg defines symbolic violence as what individuals experience when their cultural capital is devalued.

Symbolic violence is a sociological construct. In that capacity it is a powerful lens with which to examine actions of a group and ways in which certain types of knowledge are machines. This development of more complex instruction manuals and machinery necessitated a broadening shift in mathematical knowledge. Scholars even began to use more non-technical language in their written works. With the industrial age scholarly mathematics entered the school system as part of a way to ensure the education and economic dominance of a rising aristocracy. Today, amid the ideals mass education, scholarly mathematics assumes a large part of the overall curriculum (D'Ambrosio, 1985). Throughout this history, it is important to point out that scholarly mathematics consistently received more credibility and legitimacy than practical everyday ways of doing and thinking about mathematics. Although relatively more of the population has access to scholarly mathematics than in ancient times, there still exists a belief that everyday mathematical knowledge is a lesser form of *real* mathematical competence. The following section describes research efforts to unveil and legitimize such everyday mathematics and its implications for teaching and learning.

Valorization of Mathematical Practices

Valorization refers to "the social process of assigning more value to certain practices than to others" (Presmeg, 2007, p. 443). As described previously, western academic mathematics receives valorization and prominence in most school systems. However, western academic mathematics represents just one example of a culturally derived mathematical system. Many other cultures have similarly developed systems of thinking and knowing mathematics, and those mathematical processes are just as valid as those valorized in school settings.

Carraher, Carraher, and Schliemann's research in 1985, challenged the valorization of particular mathematical practices over others. Their research with young street vendors in Recife, Brazil revealed children's computational strategies different from those taught in the school. The children ranged in age from 9 to 15 years old, in schooling from one to eight years, and were very poor. Researchers (i.e., customers) interviewed each child during the course of a normal sales transaction (i.e., a real-life context), presenting various mathematical tasks. Sometimes, the actual purchase was carried out. At other times, the researcher asked the child to calculate possible purchases. Following this informal interview, the child was asked to complete a formal test comprising items with calculations identical to those problems solved during the transactions on the street.

The children's performance on mathematical problems embedded in real-life contexts (i.e., in the informal interview) far exceeded that on school-type word problems and context-free computational problems (i.e., the formal test). In fact, when given in a school-type setting, most children had no way of accessing the knowledge that they had just recently employed with the exact same numbers on the streets. Even formal test items presented as story problems did not elicit the same level of mathematical problem solving as did the transactions on the street. Clearly, the real-life context of the initial interview played a critical role in the children's access to the mathematics. Some may argue that the context gave a sense of concreteness to the task and decreased its rigor. However, all calculations on the street were carried out through mental abstractions. Therefore, the children utilized problem-solving strategies beyond the concrete.

Carraher et al. maintain that following school-prescribed routines actually interferes with problem solving. These routines tend to strip away connections to real-life and diminish mathematics to mere procedures that students may perform without understanding the underlying mathematics. Based on their findings, Carraher et al. question the claim that mathematics taught in the schools provide richer and more powerful learning opportunities. The children in this study showed more understanding of mathematical ideas in their own natural setting than

Figure 3— Probability Sort Cards

Justin flips a coin 10 times and gets 7 heads. He says the probability of a heads is 7/10.	Susan says the probability of flipping a coin and getting a heads is 1/2 because there are two sides and one is heads.	Maliik reaches into a bag of 5 marbles and picks out 3 blue marbles. He says the probability of picking a blue marble is 3/5.
Jemario uses the Probability Simulation APP to pick a marble from a bag containing 20 marbles. He replaces the marble and picks again. After 17 picks, he says the probability of picking a red marble is 17/20.	Mikayla looks at a die and says the probability of rolling a 3 is 1/6.	Leatrice rolls a die six times and gets the following numbers: 4, 5, 3, 4, 1, 1. She says the probability of rolling a 3 is 1/6.

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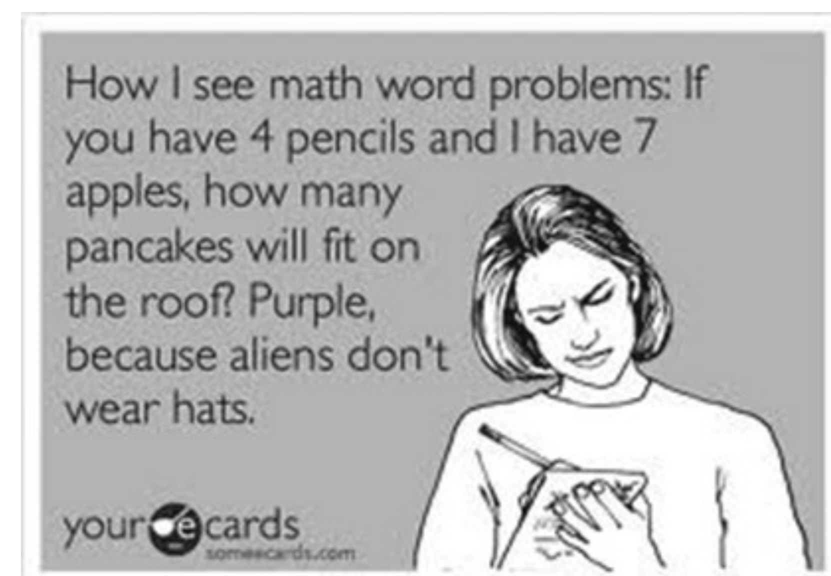


Activity Sheet 1. Predictions and Probability

Name ________________________________

MYSTERY MARBLES

A bag contains a total of thirty marbles.

There are three different colors of marbles: Color A, Color B, and Color C.

There are ten marbles of each color in the bag.

Suppose you pick a marble, write down its color, and then return it to the bag. Predict how many times a marble of each color will be picked if you do this thirty times. Remem- ber that each time you pick a marble, you return it to the bag before picking again.

Marble Color	Predicted Number	Actual Number Picked
A		
B		
C		
Total		

1. Explain how you arrived at your predictions.

2. Now use bags of marbles or the Probability Simulation APP (application) to pick a marble and re- turn it to the bag. Do this a total of thirty times. Record your results in the table above. You can use the right arrow key to see the number of marbles for each bar in the graph.

3. Do your actual results match your predictions? If not, how are they different?

4. Should your actual results match your predictions? Explain why or why not.

calculations with numbers—the way in which they learned it themselves. Recent theorists have taken a more broad view than this limited institutionalized view of mathematics. D'Ambrosio (1985) and Bishop (1988) advocated for a broadening of what counts as legitimate mathematics. Bishop identified six universal practices of mathematics: counting, locating, measuring, designing, playing, and explaining. He argues that these practices define the development of mathematical knowledge in every cultural group. Still, others have advocated for a balanced view between the formal and human aspects of mathematics (see Presmeg, 2007).

Definitions of Ethnomathematics

Ethnomathematics has emerged as a field of study that examines the relationship between culture and mathematics. However, due to the varied definitions of both culture and mathematics, there exist some difficulties in its definition. D'Ambrosio (1985), considered the father of ethnomathematics (see Appendix A), described it as on the border between cultural anthropology and the history of math.

> Mathematics is adapted and given a place as 'scholarly practical' mathematics which we will call, from now on, 'academic mathematics', i.e., the mathematics which is taught and learned in the schools. In contrast to this we will call *ethnomathematics* the mathe matics which is practiced among identifiable cultural groups…. Its identity depends l argely on focuses of interest, on motivation, and on certain codes and jargons which do not belong to the realm of academic mathematics. (italics in original text, p. 196)

In this view ethnomathematics is a dynamic and evolving system of knowledge by itself. The values and language of the particular culture determine its identity. Ascher (1991) takes a slightly different approach in defining ethnomathematics as "the study and presentation of the mathematical ideas of traditional peoples" (p. 188). This view presents it as a mathematical window on other cultures. Even though they differ in perspective, both of these definitions broaden the scope of mathematics to include marginalized everyday practices not traditionally included in "academic mathematics."

The remainder of this paper will describe briefly the development of mathematics education in Western history (a thorough historical account of all cultures' mathematical development is beyond the scope of this paper); how this development relates to the marginalization of everyday mathematical practices of non-dominant cultures; and address specific implications for classroom instruction.

Historical Overview

Historically, scholarly mathematical knowledge has received exclusive prominence in society and in the educational system. Since ancient times, mathematics has developed along two distinct paths: scholarly and practical. Ancient Greeks and Romans reserved scholarly mathematics for the select few responsible for state affairs or for tracking planetary orbits. Practical mathematics addressed the needs of manual workers or merchants. Thus upper and lower class structures were maintained, in part, by the accessibility and exclusive valorization of scholarly mathematical knowledge. In the Middle Ages, practical mathematics began to approximate scholarly mathematics with the introduction of Arabic numerals and translation of Euclid's *Elements* (D'Ambrosio, 1985). These were the first steps in making scholarly mathematics accessible to more than just the upper class.

The Renaissance period saw a change in labor structures such that laborers required a different knowledge set in order to perform their tasks. For example, bricklayers needed to interpret an architect's complex design plans; inventors needed others to help reproduce their new

Ethnomathematics: The Role of Culture in the Teaching and Learning of Mathematics

General Strand

Katie Anderson-Pence, Utah State University

In the past 30 years, mathematics education has experienced a major perceptual shift in the role of culture in the learning and teaching of mathematics. This shift in thinking has caused many researchers and educators to re-examine the historical development and practices of mathematics. Historically, many have maintained the culture-free nature of mathematics and that numbers have a universal quality about them. Can one truly divorce mathematics from culture, though? According to Zaslavsky (1996), cultural influences have a great impact on the development of mathematical thought for individual learners and for society in general. Hence, mathematics, as understood by any particular individual or society, is a cultural product. As definitions of culture and mathematics vary among academic researchers, it becomes important to explicate the meanings around which this discussion will center.

Definitions of Culture

Volumes have been written as authors have attempted to define and examine the effects of culture on all aspects of society. Stenhouse (1967) observed that culture involves the shared understandings through which individuals interact (i.e., communicate) with each other. This definition emphasizes the role of communication in culture, which has particular significance in education. In this sense, the construct of culture serves as an all-encompassing umbrella under which all human communicative activity may be examined. Culture has also been viewed as dynamic and in a constant state of transformation. This transformative view of culture underscores the importance of negotiating social norms in the classroom. Individuals transform or weave their culture through "webs of significance" (Geertz, 1973, p. 5) based on shared experience.

Culture also applies to macro-, meso-, and micro-levels within an educational sphere (i.e., society, school, classroom). The actions of teaching and learning exist in cultures that vary greatly from society to society, from school to school, and even from classroom to classroom. Nickson (1994) echoes Stenhouse's sentiments when he describes this culture as the "invisible and apparently shared meanings that teachers and pupils bring to the mathematics classroom and that govern their interaction in it" (p. 8). The negotiation of these interactions or social norms is essential as students engage in mathematical thinking in the classroom (Cobb & Yackel, 1996). Therefore, the culture of a classroom determines the type of learning that takes place and greatly affects the types of experiences in which students engage. As will be discussed below, this classroom culture can also interact with students' societal (i.e., home) culture in either positive or negative ways.

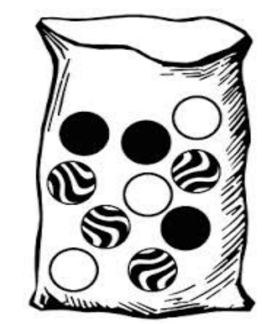
Definitions of Mathematics

Much disagreement exists within the mathematical community regarding the definition of mathematics. Generally, it has been described as "the language and science of patterns" (Steen, 1990, p. iii). Most individuals would describe mathematics as formal

Activity Sheet 2. Predictions and Probability

Name ________________________________

FIND MY MARBLES



Part 1

Assemble a bag of marbles or use the Probability Simulation APP to create a bag of marbles.

Your bag should contain a total of thirty marbles. There are three different colors of marbles. Divide your thirty marbles among the three colors however you like.

Switch calculators with your partner.

Use the Probability Simulation APP to pick thirty marbles from your partner's bag. When you have thirty marbles, fill in your results on the chart to the right. You can use the right arrow key on the calculator to see the number of marbles for each bar in the graph.

Marble Color	Predicted Number	Predicted Number in Bag	Actual Number in Bag
A			
B			
C			
Total			

1. Predict how many marbles of each color your partner has in his or her bag. Explain how you arrived at your predictions. Remember that there are thirty total marbles in the bag.

2. Now check the calculator settings to see how many of each color are in the bag. Are you surprised? Why or why not?

Part 2

Marble Color	Experimental Probability	Theoretical Probability
A		
B		
C		
Total		

Problems from *Teaching Children Mathematics*

Elementary Strand

NCTM—*Teaching Children Mathematics*

Statistics and Data Analysis

1. Your heart is a very strong muscle that works 24 hours a day, whether you are awake or asleep. Take your pulse for one minute. Use your pulse rate to determine how many times your heart beats in a day, a week, and a year. Compare your results with others' results. Create an information chart for your classmates. Research heart facts on the Internet or in your school's media center and include this information on your heart chart. Write your own mathematics heart problems and include an answer key. Ask your teacher to present your chart to the class.
2. How many hours do you and your family watch television each week? Keep a log to record the total number of hours that family members watch TV during a week. If you continue to watch TV at that week's rate, how many hours would you watch in a month? During a summer? In a year? At your weekly rate, how many hours would you watch television in one year? Too much of anything is not good for us. Log the time that you spend running and playing in the fresh air. How do the times compare?
3. Choose three or four containers of different sizes and shapes (for example, a pan, jar, glass, and so on). Put the same amount of water-for example, 1 cup-in each container. Measure and record the amount of water remaining in each container every day for a week. On the first day, predict in which container the water will evaporate first. In which one will it evaporate last? Based on your data, after the third day adjust your predictions as needed and predict when all the water will evaporate from each container.

Probability/Combinatorics

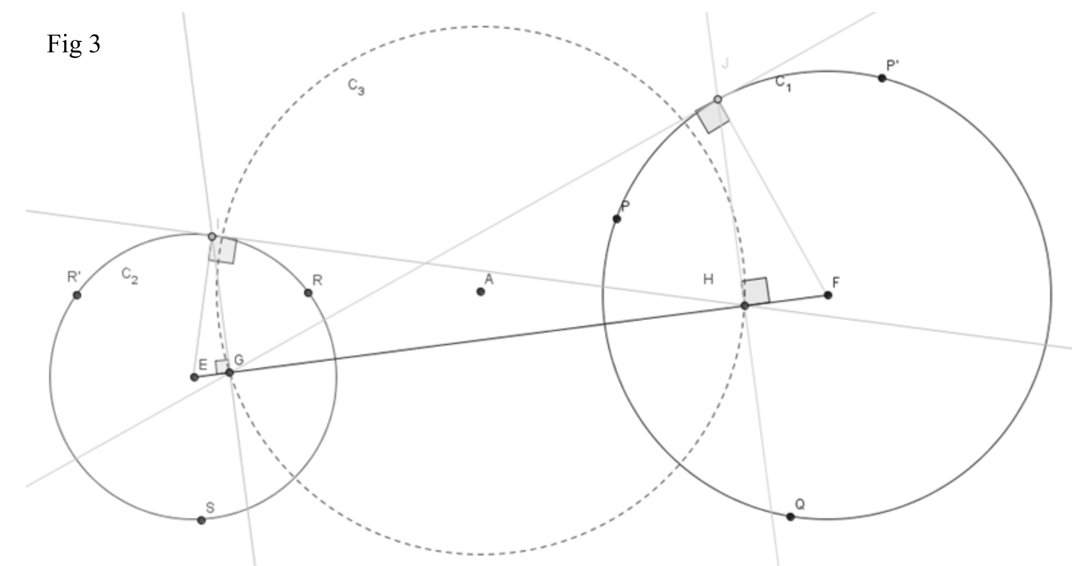
1. Write the numbers from 2 to 12 in a column. If you roll two dice and add the numbers, which sum would be most likely to occur? Roll the dice, add the numbers, and record the sum with a tally mark next to the matching number on your paper. Continue the experiment until one of the numbers has ten tally marks. Which numbers received most of the tally marks? Which number received the fewest? Why? Was your prediction correct?
2. Form a line with three other students. The first person in line shakes hands with each student in line. How many handshakes are exchanged? The next person shakes hands with all students who have not already shaken his or her hand. How many handshakes were exchanged this time? Continue until each person in line has shaken hands with everyone else.
3. How many total handshakes were exchanged by the four students? Predict and then determine how many handshakes will occur for five and six students.

How many different fractions can you write using only the digits 1, 2, 3, and 4? Be sure to include fractions greater than 1.

At this point, I was curious to see if G and H were images of each other under inversion of C_1 and inversion of C_2. To do this, I followed the same process as I did to find P' (see construction of C_1 above). As you can see from Fig 3, G and H appear to be images of each other under inversion of C_1 and C_2. Now if G and H really are images of each other, then we can use C-5.11 resulting in the conclusion that any circle containing G and H must be orthogonal to both C_1 and C_2.

As stated, G and H look like they are images of each other but it is possible that my construction is misleading regardless of utilizing GeoGebra. To prove that G and H are images of each other, we rely on the fact that C_3 is orthogonal to both C_1 and C_2. Because C_3 is orthogonal, we know that any point on C_3 must be inverted to a different point on C_3. To find out which point G maps to, construct a ray from E through G. Notice in Fig 3, combined with the knowledge that G and H lie on the line connecting E and F, H must be the image of G under inversion of C_2. Similarly, G must be the image of H under inversion of C_1. By using C-5.11, we can conclude that any circle containing both points G and H must be orthogonal to both C_1 and C_2.

Fig 3



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Note: To learn more about inversion, Read Chapter 5 Section 5 of Wallace and West (pgs. 307-316).

To find P':
1 – Construct a ray from A through P.
2 – Construct a line perpendicular to ray through P, and where the perpendicular line intersects C_3 we place and label the intersection point as C.
3 – Construct a line segment from A to C.
4 – Construct a line perpendicular to the line segment through C.
5 – Where this line intersects the ray through P is where P' lies.
6 – Create a third point Q, which is not collinear with P and P'.

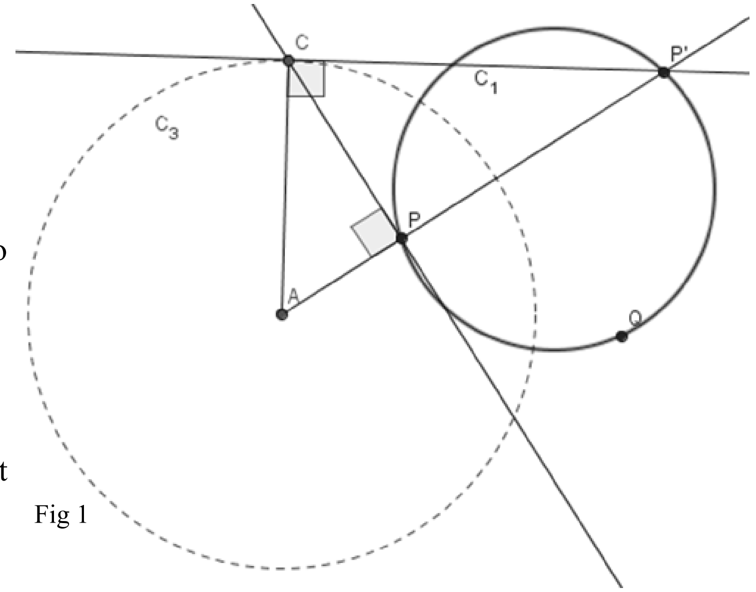
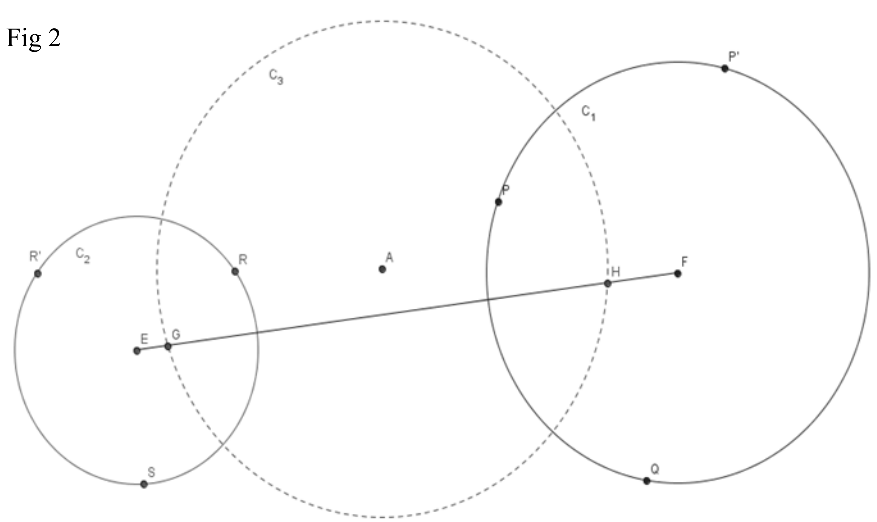


Fig 1

According to C-5.11 the circle (C_1) defined by P, P', and Q will always be orthogonal to C_3 (Fig 1). C_2 was created similarly but using different points. Creating C_1 and C_2 this way ensures that they are both orthogonal to C_3.

7 – After constructing the 3 circles, use a tool in GeoGebra to find the center of both C_1 and C_2 and label points F and E respectively. (Recall that the perpendicular bisector of any secant line of a circle contains the center point of each circle).
8 – Connect E and F, and notice that line segment EF intersected C_3 in two places. If not, then the Allred-Spencer Theorem has no use. Although if you are using GeoGebra then you can move either of the two circles so that line segment EF will intersect C_3 in two points. The two points where line segment EF intersects C_3 label as G and H (order does not matter).

Fig 2 displays the steps to this point, in the GeoGebra construction. Note, the lines, rays, points, etc… are hidden, and not erased.

Fig 2


Overcoming the "Run" Response

Middle School Strand

Patricia E. Swanson—San Jose State University

Strategies that foster self-awareness, help regulate emotions, and encourage problem-solving perseverance can turn mathematical fight or flight into re-engagement.

" This problem triggered the run response in my brain."
Amy, seventh grade

Amy's reaction to a challenging story problem reflects the anxiety that many of us face when struggling with difficult mathematics problems. Recent research suggests that it is not simply experiencing anxiety that affects mathematics performance but also how we respond to and regulate that anxiety (Lyons and Beilock 2011). Most of us have faced mathematics problems that have triggered our "run response." The issue is not whether we want to run, but rather how we ultimately turn around and re-engage with the problem.

This article examines both teachers' and students' emotional reactions to challenging mathematics problems and, more important, the strategies they use to cope with anxiety and to re-engage and grapple with these problems. These coping skills are embedded components of the first essential Standard for Mathematical Practice identified in the Common Core State Standards for Mathematics: "Make sense of problems and persevere in solving them" (CCSSI 2010, p. 6). Self-awareness and regulation are essential, and often ignored, components of mathematical problem solving. This article examines how these skills can be modeled, taught, and learned.

This work stems from a multiyear initiative designed to embed pertinent dimensions of social-emotional learning (SEL) into preservice teacher education. In selected classes, we are piloting specific strategies designed to develop both—
1. candidates' social-emotional skills for teaching; and
2. candidates' ability to foster students' social-emotional skills for learning.

Although many would view mathematics methods as an unlikely setting for this work, I suggest that teaching and learning mathematics is intertwined with the development of social and emotional learning skills essential to motivation, self-efficacy, and productive disposition toward mathematics (Kilpatrick, Swafford, and Findell 2001). These skills are particularly relevant for young adolescent learners whose social and emotional needs are closely tied to academic achievement (Bobis et al. 2011; Zollman, Smith, and Reisdorf 2011). This article will explore strategies for teaching emotional awareness and self-regulation, essential social-emotional learning skills for helping young adolescent learners engage in mathematics problems that they find difficult or even frightening.

Teacher Candidates

I began exploring self-awareness and problem solving with the preservice teacher candidates in my mathematics methods course. Their emotional reaction and self-regulation when faced with difficult mathematics problems provided the initial insights into how to pursue this work with their students. I asked teacher candidates to describe their immediate emotional reaction after reading the following problem from a middle school textbook.

Multi-Step Problem

The density of a substance is the ratio of its mass to its volume, written as a unit rate.

1. Calculate A 500 cubic centimeter sample of sea water has a mass of 514 grams. Find its density.
2. Calculate A 300 cubic centimeter sample of an iceberg has a mass of 267 grams. Find its density.
3. Compare Which is denser, sea water or an iceberg? Explain why your answer is reasonable. (Larson et al. 2008, p. 263)

Some candidates expressed confidence:

- "Excited! Love math!"
- "Love them [word problems]; way better than just digits and symbols."

Many more expressed anxiety:

- "Oh *#*; this may take a while."
- "Nervous, I'm an English major. Ahhh! . . ."
- "Fear. Math is not a subject I feel confident in doing or teaching. This problem makes me feel frustrated and disappointed in my own math skills."

The following statements were made by the anxious, not the confident, teacher candidates. It was intriguing to hear the wealth of creative ways they used to talk themselves into re-engaging with the problem. They employed a variety of self-talk (Meichenbaum 1977) strategies, internal conversations with themselves, to shape their feelings and behaviors. These strategies helped them cope with their anxiety and regulate how they would re-engage in mathematical problem solving.

- "I take a deep breath and say, 'Break it down, it's OK, you can do this.' I start to draw a picture to help myself visualize. . . ."
- " 'OK, take it one step at a time. Who cares how long it takes to solve? When you're done, double check. . . .' "
- "Reread the problem several times until [the] words and numbers start making sense. Draw/write out the problem. Work through it out loud. Make it visual so it makes sense. Ask questions."
- "Read the problem first, breathe, brainstorm, and solve what you know."

These prospective teachers recognized their stress and employed strategies to both reduce their anxiety and make sense of the math. They alluded to classic mathematics problem-solving strategies identified by Pólya (1957) including—

The Allred-Spencer Theorem

High School Strand

David Spencer, Juab Junior High School

The purpose of this paper is to clarify a theorem discovered by a colleague from my Math 3100 class (Foundations of Geometry). I have found no reference to a name in my research either online and/or other available resources. If there exists additional information I would appreciate correspondence at spenceaye@yahoo.com. The original discovery is credited to my colleague (Allred), thus the reference to the theorem, as such; Allred-Spencer Theorem.

Allred-Spencer Theorem:

Given 2 circles, C_1 and C_2, which are orthogonal to a 3^{rd} circle, C_3, such that the line connecting the center of C_1 and C_2 intersects C_3 at two points, A & B, then any circle containing both points A & B will be orthogonal to both C_1 and C_2.

I will be using a corollary from a course text book found in the works cited section, along with the definition of orthogonal.

<u>Corollary 5.5.11: (C-5.11)</u>
"If P and P' are distinct points that are images of each other under an inversion in circle O, then any circle containing both P and P' is orthogonal to circle O." (Wallace & West – Pg. 315)

<u>Orthogonal (Definition):</u>

"Two circles are said to be <u>orthogonal</u> if the radius drawn from one of the circles to a point of intersection is perpendicular, at that point, to the radius drawn from the other circle." (Wallace & West – Pg. 179 (This definition is found within exercise 37 on the mentioned page.))

In instrument used in the proof is a free downloadable geometry program called GeoGebra. GeoGebra. My process began by utilizing the GeoGebra program to create two circles orthogonal to a third circle. To duplicate the process, I will outline my steps. First, create and label the third circle (C_3), and find the image of a point P (P'), in which point P lies inside of C_3.

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1. understand the problem (i.e., "draw a picture to help myself visualize");
2. choose a strategy (i.e., "brainstorm, and solve what you know");
3. follow the strategy (i.e., "work through it out loud"); and
4. evaluate the strategy (i.e., " 'when you're done, double check' ").

To cope with their anxiety, they re- minded themselves to slow down and breathe. To re-engage, they searched for and used known aspects of the problem. Their comments indicated a strategic integration of emotional self-awareness, coping, and problem- solving skills—all attributes that are well worth modeling and teaching to students.

Self-Awareness in Middle school students

The density problem was posed to an academically diverse group of sixth- grade through eighth-grade students in a rural school. I asked the students to describe their emotional reaction to the same problem. Although I knew that they had been exposed to the necessary math, their reactions mirrored those of my less confident teachers:

- "It triggered the run response in my brain." (Amy)
- "This doesn't make any sense to me. I'm confused with the first sentence. I just want to forget about it." (Mike)
- "It is really hard, and I would feel terrified of failing it. I haven't been taught this." (Gloria)
- "I can't do it. I do not know what to do." (Cecilia)
- "It looks hard and complicated." (Juan)
- "I begin to freak out as I read and reread the problem." (Cory)
- "What the heck are they talking about?" (Bill)

Some felt anger, and others appeared hopeless. Most felt some degree of fear. I asked them what they would say to themselves to cope with their feelings and talk themselves through the problem. Most looked baffled by the question and simply said they do not talk themselves through problems. They stop working or ask for help. Only one student, Cory, said he would reread the problem and look for parts he knew.

I showed them the kinds of strategies my teacher candidates used, but first I let them read some of my prospective teachers' reactions to the problem. They were amazed and highly entertained that teachers' emotional reactions to difficult problems could mirror their own. We talked about recognizing how a problem might scare them and how to calm down and take their time. These self-talk strategies acknowledged and addressed students' emotional and physical reactions to the problem. We then practiced problem-solving self- talk for re-engaging and addressing the cognitive demands of the problem.

We studied the picture of an iceberg that appeared next to the problem in the textbook and decided to try Cory's strategy and "reread the problem." I read it aloud to allow struggling readers to think about the problem. Although they were "freaked out" at the academic language, stu- dents began to pick out words they knew ("solve what you know"). Mike knew about volume and mass and gave a credible description of density as the size of something in relation to its weight. We discussed the density of their math book compared with the density of a stuffed animal that is in the school's reading center. Gloria commented that ice floats. The group pondered why and how that might be related to the density of the iceberg. They proposed that since icebergs float, they should be less dense than seawater.

I commented that students had essentially answered the problem by combining what they knew with what they figured out without ever doing any math. They exchanged the smug looks of students who had beat the system.

Nonetheless, we decided to engage in the math by trying the step-by-step approach next and built the ratio for density, density = mass/volume, from the sentence of the problem. I helped stu- dents link mass to grams and volume to cubic centimeters, and they built the ratio for the density of seawater, d = 514 g/500 cc. They eagerly set up the iceberg ratio, d = 267 g/300 cc, on their own. Amy noted that the sea- water ratio was more than 1, whereas the iceberg ratio was less than 1, so seawater must be more dense than the iceberg. They had essentially solved the last part of the problem by acknowledging and coping with their emotional reaction, re-engaging, and taking time to make sense of the math. Mike, who initially wanted to "just forget it," insisted on staying to calculate the unit rates and prove that they were correct. I asked the students to reflect on their initial reaction to the problem and what they had learned.

- "If I paid attention to what I knew instead of freaking out, I would have actually gotten the problem." (Amy)
- "It is quite simple when you calm down." (Gloria)
- "I could have done this really quickly if I wasn't freaking out." (Mike)

Students reflected that the math was actually easy. One student commented that "it was just all the words mixed together" that made the problem difficult. They were glad they had not had to tackle it alone and commented that they needed the teacher's help to understand the problem and help them find the parts they knew. However, when I asked them if they could imagine talking themselves through a problem like this on their own at some point, they agreed that with practice, they probably could use the coping and problem-solving strategies we had tried that day.

Lessons Learned

This case provides insights into how self-awareness and problem solving may be taught. First, it describes a lesson that could be aligned with teachers' content objectives at any point in a mathematics curriculum. The lesson could be introduced or revisited whenever students face a potentially intimidating problem-solving task. Teachers should choose the task carefully, finding a problem that they suspect students will find intimidating but that will require skills most students possess and that will offer multiple entry points and solution paths. Within this context, the lesson described in this case is essentially a discursive frame including three steps: recognizing and acknowledging emotional reactions, developing self-talk and coping strategies, and providing cognitive scaffolding during the problem-solving process.

Acknowledging Emotional Reactions

It is essential to begin the lesson by asking students to describe their emotional reaction to the problem. **Figure 1** provides some of the prompts that teachers could use to help students recognize their emotional reaction and its impact on how they approach (or avoid) the problem.

In my many years of teaching mathematics, I had never explicitly asked students how a problem made them feel. Prompting both groups— teachers and students—to examine their

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in discussion-oriented classroom communities are sharing and listening. First, students must take responsibility for sharing the results of their explorations and for explaining and justifying their strategies. Second, students must realize that learning means learning from others, taking advantage of others' ideas, and listening to the results of their classmates' investigations (Hiebert et al., 1997). Thus, to become full participants in a community of peers doing mathematics, students must be willing to share with and actively listen to one another.

Research by Otten et al. (2011) suggested that when students actively listen to one another, mathematical reasoning can be made more explicit and more accessible. As a result, more students can participate in the discussion by articulating mathematical thoughts and developing shared meanings. This type of community knowledge-building can cause students to compare and contrast their own mathematical thinking to that of their peers, change their own thinking, and come to new understandings (Kosko, 2012). The teacher plays an important role in helping students understand what counts as an acceptable explanation and justification in mathematics class (Yackel & Cobb, 1996) so that students' efforts to listen to each other are not hampered by student talk that is unclear or imprecise.

Concluding Thoughts

The fact that the rules of the IRE pattern, the defining characteristic of recitation, so heavily favor the power of the teacher is undoubtedly one reason why it has become such a popular style of teaching (Lemke, 1990). Teachers understandably may find it difficult to deviate from IRE because maintaining it offers many advantages to them, such as setting the topic, controlling the pace, and steering the direction that the topic develops (Lemke, 1990). Thus, navigating a new terrain of teaching can be challenging for teachers at any level, particularly because they may never have experienced, as a learner, an approach to teaching other than lecture or recitation (Marrongelle & Rasmussen, 2008). Some teachers have handled this challenge by believing they need to stop all "telling" (see, e.g., Chazan & Ball, 1999). Yet, the recitation versus discussion interaction patterns need not be dichotomous. Acknowledging that talk formats operate on a continuum, some researchers have pointed out that most classrooms operate somewhere between recitation and discussion (Herbel-Eisen- mann, 2001). Cazden (1988) contended that within a matter of moments, a lesson can move from recitation to discussion, and the activity that students are engaged in can determine the form of the lesson. As a general rule, however, any extreme version of the IRE-recitation sequence can be viewed as having the potential for closing down the discourse. In contrast, as teachers move away from recitation toward more purposeful discussions, there is a potential for opening up the discourse and shifting the mathematical authority from teacher to community.

To be clear, it is not just getting students to talk more that matters. The orchestration of the discourse must be purposeful (Smith & Stein, 2011), and it must be academically productive "in that it supports the development of students' reasoning and students' abilities to express their thoughts clearly" (Chapin & O'Connor, 2007, p. 115). The field is just beginning to understand and develop ways to support teachers in facilitating productive discussions in mathematics class. More studies are needed that validate the effectiveness of some of the existing strategies available for orchestrating productive discussions. In addition, the field would benefit from studies that identify features of unproductive discussions that inhibit student learning.

Acknowledgments

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emotional reaction to a problem provided space to discuss what we usually keep hidden. For the many students who reacted with anxiety, the discussion created a sense of safety in numbers as they heard similar sentiments from so many of their peers.

It was important to emphasize that feeling an emotional reaction to the problem was not wrong. Recognizing one's feelings and learning to cope with them was the objective. Describing their emotions paved the way for students to try the coping strategies and self-talk described in the following section.

Figure 1: These prompts promote students' self-reflection.

Recognizing Emotions and Their Impact

1. How did this problem make you feel?
2. What did you say to yourself when you first read this problem?
3. How do our emotions and beliefs influence what we choose to do?

Developing Self-Talk and Re-Engage Strategies

With both teachers and students, we brainstormed self-talk strategies after discussing their emotional reaction to the problem, both to relieve stress as well as to re-engage with the problem.

Prospective teachers in my class had a wealth of such strategies, implying that they had considerable experience talking themselves through both their anxiety and the mathematics. Students' repertoire of self-talk was far less developed, a finding that speaks to the need to develop students' coping strategies in mathematics. When faced with a difficult mathematics problem, many students do not try to figure it out. They simply stop and wait for the teacher to tell them what to do. **Figure 2** provides examples of the kinds of self-talk that students could use to calm themselves and re-engage with difficult problems.

It is important to have students brainstorm ideas first, chart them, and try them, giving credit to the students who generated the ideas. Ask students to think about a time when they experienced a problem outside of the context of math. How did they overcome it? What kinds of things did they say to themselves in the process? These questions provide ways to start the conversation. If students are unable to suggest any coping strategies, it is important for the teacher to be ready with suggestions, thereby modeling self-talk both for reducing anxiety (take a deep breath, relax, take your time) and for re-engaging with the problem (reread it, find what you know, work step by step).

Alternatively, the teacher could use the preservice candidates' emotional reactions and self-talk described in this article as a discussion starter for this segment of the lesson. In the lesson described above, reflecting with the students on prospective teachers' emotional reactions to the problem and the self-talk that candidates generated moved the discussion forward. I believe that students felt that if successful college graduates—future teachers—were anxious when faced with a challenging math problem, then surely it was OK for them to "freak out." As my middle-grades students shared the same anxiety as my teacher candidates, they were willing to try some of the same coping strategies.

Figure 2: These self-talk strategies provide re-engagement entry points.

Self-Talk and Coping Strategies

1. Take a deep breath and relax.
2. Take your time to re-engage.
3. Reread the problem and find what you know.
4. Take it one step at a time.

Providing Cognitive Scaffolding

Supportive scaffolding took place throughout the lesson. In the first two steps of the lesson, the scaffolding that helped students recognize and cope with their anxiety cleared the way so that students could cognitively engage with the problem. In the third and final step, cognitive scaffolding helped them make sense of the problem and find places in the problem where their prior knowledge or experience could help them. I read the problem aloud.

We looked for words or scientific ideas (e.g., density) that at least some of the students knew. I gave positive specific feedback validating students' knowledge. Only once did I provide clarification: I stepped in to help them link volume and mass to their measurements in cubic centimeters and grams.

We discussed the picture of the ice- berg and searched for clues in the text. Students shared what they knew and pieced their understanding together like the parts of a puzzle. I did not, however, tell the students how to do the problem. I helped them become aware of their feelings (step 1), cope with those feelings and re-engage (step 2), and persist in solving the problem (step 3) by making sense of it and piecing together what they already knew. **Figure 3** identifies specific cognitive-scaffolding strategies that can be used to help students solve the problem themselves.

Figure 3: These cognitive-scaffolding strategies will promote moving toward a solution.

Cognitive Scaffolding

1. Assist struggling readers by reading the problem aloud.
2. Help students use context clues in the text and pictures to make sense of the problem.
3. Provide positive specific feed-back validating both students' knowledge and their use of self- talk or coping strategies.
4. Honor incremental advances in solving the problem and multiple solution paths.
5. Do not tell students how to do the problem; validate their effective strategies.

The Value of Emotional Awareness

This case provides an instance of the intersection of mathematics and social-emotional learning skills. It demonstrates how self-awareness and problem solving interact when doing mathematics. It is a case that has changed my practice. I do not believe that I had ever before asked either adults or students how a mathematics problem made them feel. When I did, the range and depth of their emotions surprised me. As my students discussed their emotional reactions, they learned that they were not alone in their feelings.

Helping students develop the skills to recognize and regulate their emotional reaction set the stage for re-engaging with the problem and making sense of it. The discussion gave students the coping skills necessary to persist. Sense-making and persistence are foundational mathematical practices. Emotional awareness and regulation helped students engage in these practices. This lesson was time well spent. Self-awareness and problem solving are essential not only for mathematics but also for life.

"Knowledge is built. Understanding grows. Relationships with mathematics and with classroom community members develop" (Middleton & Jansen, 2011, p. 164).

The case of Railside High School also offers evidence that students' motivation to learn mathematics can be positively impacted by participating in discussion-focused classrooms. The results of questionnaires given to students showed that each year the Railside students were significantly more positive about their mathematics experiences than their peers in more traditional classes. For example, 71% of Railside students in Year 2 classes (n = 198), reported "enjoying math class," while only 46% of students in the more traditional classes (n = 318) agreed to this statement (Boaler & Staples, 2008). By their senior year, 41% of Railside students were in advanced classes of precalculus and calculus, compared to only 23% of students coming from the more traditional class- es (Boaler, 2008).

Discussion Can Support Teachers in Understanding and Assessing Student Thinking

Some classroom interaction patterns promote deeper mathematical thinking than others (Herbel-Eisenmann & Breyfogle, 2005; Martens, 1999), and skillful questioning of student thinking can provide the teacher with valuable knowledge about students' developing mathematical ideas (Martino & Maher, 1999). NCTM's (2000) Teaching Principle begins with the following claim: "Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well" (p. 16). Discussion is a strategy that can support teachers in understanding what students already know and in determining what they still need to learn. In this sense, listening to students' ideas in discussions can serve as formative assessment that helps teachers make decisions about instruction. To maximize the instructional value of discussion using formative assessment, "teachers need to move beyond a superficial 'right or wrong' analysis of tasks to a focus on how students are thinking about the tasks" (NCTM, 2000, p. 24). Rather than concentrating solely on misconceptions or errors, teachers should make efforts to identify valuable student insights on which further progress can be based (NCTM, 2000). Emphasizing tasks that focus on reasoning and sense-making and providing students with opportunities to discuss mathematics serves to afford teachers with ongoing assessment information. Teachers must then guide the students toward new understandings and support their development as they work to communicate mathematically.

A key component of formative assessment is feedback. When students routinely take part in discourse in which meanings are developed and shared, they are provided with feedback that supports them to move their learning forward (Lee, 2006). In particular, feedback allows students to compare how their thinking correlates with that of other students in the class as well as the conventional mathematical ideas. It also allows students opportunities to reconsider and revise their thinking from the early "first draft" stage to a more re- fined "final" version (Choppin, 2007). A discussion-rich learning environment can provide students with agency over their own learning.

Discussion Can Shift the Mathematical Authority to Community

When teachers shape the discourse by opening it up through discussion, there is real potential to shift the mathematical authority from teacher (or textbook) to community (Webel, 2010). For this shift to truly be realized, however, the students must be aware of and willing to take on roles that differ from their roles in recitation sequences. More specifically, for discussions to be productive, students must "share the responsibility for developing a community of learners in which they participate" (Hiebert et al., 1997, p. 16). Two important aspects of the students' role an urban school in California, the focus of the approach to teaching mathematics was "communicative," meaning that "the students learned about the different ways that mathematics could be communicated through words, diagrams, tables, symbols, objects, and graphs" (Boaler, 2008, p. 59). As they worked on algebra and geometry tasks in heterogeneous classes, the students would frequently be asked to explain their work to each other. In fact, teachers lectured only about 4% of the time. Approximately 72% of the time, students worked in groups while the teachers circulated around their rooms showing methods to students, helping students, and asking them questions about their work. Students presented their work about 9% of the time, and they were questioned by the teacher in a whole-class format about 9% of the time (Boaler & Staples, 2008). As part of the research project, the achievement of Railside students was compared to that of similar-size groups of students being taught through more traditional approaches in two other high schools. In these classes, students did not typically discuss mathematics, but rather watched the teacher demonstrate procedures and then worked through textbook exercises. At the beginning of the year, the two suburban schools using the more traditional approach started with higher mathematics achievement levels than the students at Railside, but by the end of the first year of the study the students at Railside were achieving at the same level in algebra as the students in the suburban schools. By the end of the second year, the Railside students were outperforming the other students on algebra and geometry tests (Boaler, 2008).

One more piece of evidence to support the idea that dis- cussing mathematics can lead to increased student learning comes from a study focused on students' perspectives. In Listening to My Students' Thoughts on Mathematics Education, mathematics teacher Joseph Obrycki (2009) described the results of his action research project in which he analyzed six interviews of students in his high school geometry course. The interviews were conducted for Obrycki by a university researcher after he participated in three years of professional development focused on classroom discourse. Obrycki's students noted again and again that his teaching style was different from their past mathematics instructors (who told them about mathematical ideas) because he expected them to think and "figure stuff out" themselves. Some students noted an initial frustration with this approach, but eventually all students interviewed concluded that working in groups to prove theorems and solve problems was in their best interest in terms of their learning. All six students agreed that it was possible to generate mathematical knowledge on their own, with many noting that this was the best way to learn. When asked at the conclusion of the interview if there was anything she would like to share with other mathematics educators, one student noted: "I don't know if the answer should be withheld all the time, but letting students get to the answer and not just presenting it to them is definitely worthwhile, even if it takes longer" (Obrycki, 2009, p. 201). When students begin to recognize that participating in mathematics discussions helps them to learn mathematics, their motivation to participate may be increased.

Discussion Can Motivate Students

In Motivation Matters and Interest Counts, Middleton and Jansen (2011) suggested that teachers should make efforts to involve their students in class by convincing them that many types of contributions will help advance the class's knowledge (e.g., questions, alternative solutions, false starts, conjectures). When teachers do this, they argued, more students feel comfortable and courageous enough to contribute to classroom discussions. Active participation in a collaborative mathematics classroom, therefore, can have a positive impact on student motivation:

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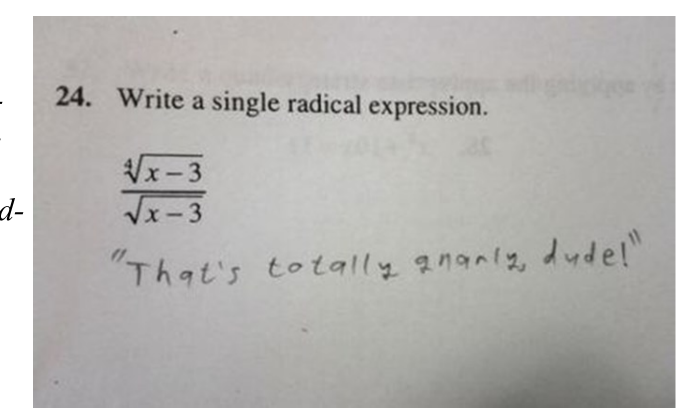

Beyond Logic: Tapping into the Multiple Intelligences in the Mathematics Classroom

Middle School Strand

Melissa Cossey, Utah Valley University
Suzy Cox, PhD, Utah Valley University

Abstract

Mathematics scores are stagnating at the eighth grade level in the United States, revealing that student performance is not improving at the national and international levels. Changes in mathematics instructional practices are necessary to engage students and demonstrate the relevancy and application of mathematics beyond the classroom. Howard Gardner's theory of Multiple Intelligences may provide one direction for instruction and assessment to jump-start mathematics achievement and help students develop greater interest in and motivation toward the subject.

Introduction

There is a growing trend in the United States of students who graduate from high school hating math or, if not hating it, seeing no application of it beyond the classroom. This mindset can continue through college, with many students taking only the required math for their lower division general requirements, or taking the required courses in high school to avoid taking math at all as part of their collegiate degree. This mindset does not spontaneously occur in students. For secondary educators, the eighth grade year appears to be of most concern as eighth grade mathematics test scores in the United States have stagnated in the past several years (National Center for Educational Statistics, 2011b). This is concerning because eighth grade is a critical year, often predicting students' achievement and trajectories as they continue on to high school, where grades begin to affect the outcome of where they will end up after graduation. For students who graduate from eighth grade not understanding math, high school math and college math become next to impossible.

Therefore, we must evaluate how mathematics is taught. In an age of unparalleled diversity and with a student body with an incomparable desire for relevance, mathematics instruction cannot remain as mainly lecture with carefully contextualized application problems that only require the student to apply one specific skill. Rather, mathematics instruction must be designed in such a way that application is relevant and significant to students, showing not only how math happens in the real world, but how it can be uniquely applied in the situation of every student. Additionally, we must recognize the diversity of students in our classrooms and design instruction that better reaches all learners.

The Problem

Students see little application for higher-level mathematics in their future careers and, therefore, many students are unmotivated to pursue mathematics education. Yet mathematics plays a significant role in whether or not a student goes to college. It is reported that the current instructional approach taken in middle school math "denies some students

Here, Ms. D seemed genuinely curious about how students were making sense of the problem. She was interested in learning about the range of solutions, and she allowed misconceptions to surface. Although the teacher was orchestrating the discourse, students were encouraged to speak with one another in the discussion. Ms. D also provided students with opportunities to use mathematically precise language and to engage with the reasoning of their classmates. *Attending to precision* by communicating precisely to others and *constructing viable arguments and critiquing the reasoning of others* are two practices that are promoted in the Common Core State Standards. These practices are based on the belief that students learn and come to understand mathematics by working to justify why a mathematical statement is true or where a rule comes from (NGA Center and CCSSO, 2010). In the sections that follow, some benefits of discussion are described.

Discussion Can Increase Student Learning

The classroom culture, the ways in which students and teachers interact, the kinds of learning experiences students have, and the tasks that students are asked to engage with all greatly influence the opportunities that students have to learn mathematics in any given classroom (Hiebert et al., 1997). We learn through social interaction (Lave & Wenger, 1991; Vygotsky, 1978). A Vygotskian viewpoint, as articulated by Gibbons (2006), suggests that language use is at the root of learning. More specifically, this view of language calls for any examination of teaching and learning to treat interactions between teacher and learner as crucial. These interactions not only shape students' talk, but they help to construct understanding (Gibbons, 2006). Discussions can take place in small groups or as a whole class. When viewing a classroom as a community of learners, it must be remembered that interacting is not optional, but rather it is essential because communication is necessary for building understanding (Hiebert et al., 1996). In the remainder of this section, three studies which support the idea that discussion-based classrooms can increase student learning are summarized.

The results of Project Challenge offer compelling evidence that shifting to a discussion-based teaching format positively impacts student learning. In their work with Project Challenge, Chapin, O'Connor, and Anderson (2003) put a great deal of emphasis on students talking with one another and with the teacher in particular ways that have been found to be academically productive. The work of Project Challenge took place over four years in a low-income Boston school district and involved about 400 students and 18 teachers in grades 4–7. The majority of these students (65%) were English Language Learners, and most students (78%) qualified for free and reduced lunch. Using Standards-based curricula, daily logic-problem warm-ups, and weekly quizzes, these classrooms "emphasized communication by supporting discussions, both lengthy and brief, and by maintaining a constant focus on explanations for students' reasoning" (Chapin & O'Connor, 2007, p. 114). Results on the California Achievement Test (CAT) were used as a measure of student learning. After about three years of the study, the class mean of the Project Challenge students reached the 90th percentile. Project Challenge students also scored better as a whole than students in one of the most highly ranked cities in the state of Massachusetts (see Chapin & O'Connor, 2004; 2007 for more details). These results provide strong evidence that student learning is greatly supported by engagement in academically productive talk (Chapin & O'Connor, 2007).

The case of Railside also suggests that students learn more in classrooms that provide them with opportunities to learn mathematics through discussion. At Railside High School, *Ms. R:* Good, two. And what were those two points?

Jamie: One, six and, um, six, eleven.

Ms. R: Good. The intersection points are one, six and six, eleven. Let's look at another one.

In this recitation sequence, Ms. R seemed to be looking for correct answers. She did not appear to be focused on understanding her students' thinking or providing opportunities to discuss strategies using mathematical language. One of the most striking features of a typical recitation sequence is that the teacher tends to be the only one asking questions, as seen above. Thus, recitation could foster the impression that students must participate in accordance with the pattern established by the teacher—namely, students speak only when invited to respond to their teachers' questions.

Discussion provides an alternative to recitation. Within discussions, assessing students' subject-matter knowledge is not necessarily the primary and sole objective. In addition, teachers are interested in helping their students to develop understandings. In the example below, Ms. D works on the same problem as Ms. R, but this time through a discussion rather than a recitation.

Ms. D: Okay, let's talk about the next problem. You were asked to figure out something about the points of intersection of the parabola and the line. What did people come up with?

Jen: We said there was one point.

Juan: My group got two.

Maria: Yeah, we got two too.

Ms. D: All right then, let's take a look at this. I'm hearing that some groups found that there was one point of intersection and others thought that there were two. Let's hear from Maria's group first. Maria, can you describe your strategy?

Maria: Well, we just graphed the parabola and the line, and then we found that they intersected at one, six and at six, eleven.

Ms. D: You graphed it how?

Maria: We used our graphing calculator. At first we thought that there was one point too, and then we had to change the screen and we found the second point.

Ms. D: Does anyone have questions for Maria?

Jen: What do you mean you changed your screen? Because we graphed ours too.

Maria: We had to change the numbers so we could see the graph bigger. Then we saw the two points when we changed to bigger numbers.

Ms. D: Does anyone understand what Maria is saying about seeing the graph bigger and changing to bigger numbers? Can anyone else restate what Maria said using some of the terminology that we discussed yesterday? Grady?

Grady: Yeah, I think she's saying that she changed her viewing window. She probably had to change the y-values so she could see the graph higher. That's what we did because if you just use the normal window then you can only see one point. But we knew there had to be two points because we talked about how if there's only one point, it goes along the side of the graph.

Ms. D: Okay, so I think what you're saying at the end is that if there was only one point of intersection, it would have to be a tangent line, tangent to the parabola. [Ms. D draws a diagram on the board.] Is that what you're saying, Grady?

Grady: Yeah.

Ms. D: Jen's group—did what Maria and Grady said make sense?

Jen's group: Yeah.

access to higher mathematics" (Center for the Study of Mathematical Curriculum, 2004, p. 5) implying that for some students, middle school mathematics does not provide a lasting application nor instill the need for higher mathematics once that student reaches higher education. Mathematics scores are not improving at the middle school level, leading to students not pursuing higher mathematics in high school and college, yet students need to use mathematical problem solving skills throughout their lives. A new approach to mathematics instruction is needed to create relevancy, interest, and an environment for success, with the goal of encouraging students to pursue higher-level mathematics courses in high school and college.

Using Gardner's Multiple Intelligences (MI) theory as a guide, mathematics can become not only more engaging, but relevant and applicable to students as well. This has to start at the instructional design level by developing methods for "mobilizing the Multiple Intelligences to achieve specific pedagogical goals" (Gardner, 2011, p. xxi) in the mathematics classroom. An instructional foundation in Multiple Intelligences can bring mathematics to life. Additionally, using the Multiple Intelligences as a framework for instruction can help math educators to reach the diversity of learners in our classes.

Mathematics Achievement in the United States

In *The Nation's Report Card*, a document "[informing] the public about the academic achievement of elementary and secondary students in the United States" (National Center for Educational Statistics, 2011a, p. 0), findings about mathematical achievement across the fourth and eighth grade years were published. The results of the report card "are based on nationally representative samples of 209,000 fourth-graders from 8,500 schools, and 175,200 eight-graders from 7,610 schools" (NCES, 2011a, p. 6). Students' performance was measured at these grade levels in the following areas: number properties and operations, measurement, geometry, data analysis, statistics/probability, and algebra. Furthermore, the distribution of each of the five categories was also presented in different percentage allotments between the grades based on mathematical learning that occurs at each grade level. Table 1 shows how the test was weighted according to content matter.

Table 1. Target percentage distribution of NAEP mathematics questions, by grade and content area: 2011

Content Area	4th Grade	8th Grade
Number properties and operations	40 %	20 %
Measurement	20 %	15 %
Geometry	15 %	20 %
Data analysis, statistics, and probability	10 %	15 %
Algebra	15 %	30 %

(NCES, 2011a, p. 6).

As shown, the fourth grade assessment focuses on the students' ability to work with number properties and operations while the eighth grade assessment places stronger emphasis on Algebra and other higher-order processes.

The Nation's Report Card shows the scores of eighth graders going up by one point overall from 2009 to 2011. However, an examination of each of the states' scores reveals that in 36 states there were no significant changes in the math scores of 8th grade students during this period (NCES, 2011a, p. 47). This trend suggests that there is stagnation in eighth grade math scores in most of the United States.

Furthermore, the National Center for Educational Statistics, part of the US Department of Education, also conducted the *Trends in Mathematical and Science Study (TIMSS),* most recently in 2011, to examine US mathematics achievement compared to other countries. The mean score on the *TIMSS* is 500. This study found that "[t]here was no measureable difference between the US average mathematics score at grade 8 in 2007 (508) and in 2011 (509)," revealing that US eighth grade scores have been stagnant since 2007. In looking at the international benchmarks presented in the *TIMSS*, among US eighth graders, only seven percent reached the advanced international benchmark- which is a score of 625 (National Center for Educational Statistics, 2011b).

The scores presented in the *Nation's Report Card* and *TIMSS*, show that eighth grade achievement in mathematics in the US is not improving. Yet eighth grade is a critical year in determining whether or not students continue in higher-level mathematics in high school and college. Richard Riley, former US Secretary of Education, in the *Executive Summary of Mathematics Equals Opportunity,* stated:

> The eighth grade is a critical point in mathematics education. Achievement at that stage clears the way for students to take rigorous high school mathematics and science courses—keys to college entrance and success in the labor force. However, most eighth and ninth graders lag so far behind in their course taking that getting on the road to college is a long way off (Riley, 1998).

This situation creates a dilemma, not just for students and teachers, but the country at large. If students are not striving for higher mathematics because they decided in eighth grade that math was "too hard" or "too boring" or not applicable now or for their future ambitions, then emphasis must be placed on improving mathematics instruction and learning to foster an interest in mathematics at that level.

The report from Riley also indicated that, "Making a successful transition from arithmetic to more advanced mathematics, including algebra and geometry, has often been difficult for students" (Riley, 1998). This is, perhaps, based in part on the past curricular practices of the US education system, in which algebraic and geometric ideas have not typically been presented in full until upper grades, making the content new for students at a time when they are still struggling to move from concrete to abstract conceptualizations cognitively. Students in the middle school years are generally functioning in the concrete-operational stage of cognitive development according to Jean Piaget's theory, meaning that they typically use "hands-on thinking" and require concrete examples that are often experienced rather than visualized and conceptualized abstractly (Woolfolk, 2011, p. 49).

There are also critical implications for the outlook of scores in mathematics and students' ability to succeed in and pursue higher education in general. For instance, Riley reported:

> Data from the National Educational Longitudinal Study (NELS) reveal that 83 percent of students who took algebra I and geometry went on to college within two years of their scheduled high school graduation. Only 36 percent of students who did not take

What Does Research Say the Benefits of Discussion in Mathematics Class Are?

High School Strand

Michelle Cirillo, University of Delaware

NCTM Discussion Research Brief, 2013

Introducing new material in mathematics class in the United States has typically been done through teacher presentations of a few sample problems followed by demonstrations of how to solve them. The step-by-step demonstrations are often carried out by asking short-answer questions of students along the way (Stigler & Hiebert, 1999). Over the last 20 years, however, mathematics educators have observed and analyzed alternatives to recitation, the questioning pattern described above. In particular, a growing body of literature supports the use of discussion in mathematics class. In this brief, after describing and providing examples of recitation and discussion, some benefits of discussion in mathematics class will be presented. These recommendations are based on published studies that suggest that discussion is a productive alternative to other more passive talk formats. In short, discussion can:

- Increase student learning
- Motivate students
- Support teachers in understanding and assessing student thinking
- Shift the mathematical authority from teacher (or textbook) to community

Recitation versus Discussion

In *Classroom Discourse*, Cazden (2001) made the following observation: The three-part sequence of teacher Initiation, student Response, and teacher Evaluation (IRE) is the most common pattern of classroom discourse at all grade levels. The IRE interaction pattern repeats itself throughout a recitation-type lesson. In their succinct summary of implicit rules, Edwards and Mercer (1987) noted: (a) It is the teacher who asks the questions; (b) The teacher knows the answers; and (c) Repeated questions imply wrong answers (p. 45). Below is an example of a recitation sequence contained in a lesson in which Ms. R is working with students on the problem of finding the points of intersection of a line and a parabola:

Ms. R: Let's look at the third problem. How many points of intersection did you come up with? Chris?
Chris: One.
Ms. R: One? Jamie?
Jamie: Uh, two.

(This article is reprinted with the permission from the National Council of Teachers of Mathematics. This article appears in the Discussion Research Brief, 2013)

Notes Page

During Math Test
My answer: 28
Answer choices:
17, 19, 26, 36.
Me: "well 26 is closer to 28, so that must be the answer."

algebra I and geometry courses went to college (1998).
Riley also indicated that higher percentages of students from low and middle income families that took these math courses went to college than those in similar circumstances who did not take these math classes (1998). These figures communicate a need for improved mathematics education as students who feel competent in the subject areas of algebra and geometry are more likely to pursue higher education at the collegiate level, yet national testing reveals students are not improving on mathematics tests, particularly in these advanced areas.

Traditional Pedagogies in Mathematics

While the scores show little advancement in mathematical achievement, rigor of the content is not the only factor in determining whether students fail or succeed. Instructional approach also plays a key role in learning outcomes. The National Council of Teachers of Mathematics (NCTM) advocates for new focus in mathematics instruction in which students are encouraged to wrestle with problems rather than simply use rote memorization of formulas to compute answers. The NCTM has argued for changing instruction to allow students to problem solve and see application for mathematics in a relevant context. Students at the middle school level also function at a cognitive level in which hands on activities in multiple instructional approaches results in greater understanding and achievement (Center for the Study of Mathematics Curriculum, 2004,p. 6). The promotion of these changes suggests that the current practices of mathematics teaching are largely lectured-based, where students memorize formulas to apply in specific situations, not allowing students to see the broader applications beyond the assigned problem sets. The predominance of lecture-based instruction is also challenging for many students who have difficulty learning from this format. The implication is that mathematics instruction cannot be "thirty minutes… spent reviewing the previous day's lesson, ten minutes… teaching new material, and the last five minutes on the students' working the problems" (Martin,1996, p.1). Instead, mathematics instruction should be delivered in a variety of formats, encompass a breadth of problems found in life, and provide ways for diverse students to rely on mathematics principles to uniquely approach problem solving. Howard Gardner's Theory of Multiple Intelligences is one framework that can help teachers think about structuring mathematics instruction to meet these needs.

Gardner's Theory of Multiple Intelligences

While there are many ways in which intelligence can be viewed and defined, Harvard psychologist Howard Gardner's Theory of Multiple Intelligences (MI) views intelligence in a multifaceted sphere, in which each human being contains not just one realm of intellectual capacity, or general intelligence, but, rather, a conglomeration of intelligences that serve an individual in context.

Gardner established that in order for something to be defined as an intelligence it had to enable "the individual to resolve genuine problems or difficulties" and furthermore "entail the potential for finding or creating problems—thereby laying the groundwork for the acquisition of new knowledge" all while proving to be "of some importance within a cultural context" (Gardner, 2011, p. 64-65). Simply put, intelligences are "semi-independent ways of 'solving problems and fashioning products'" (Wahl, 1999, p. 2). In this respect intelligence must be useful for solving the myriad of problems that individuals encounter within their realm of development and culturally associated reality and create a relevant solution within that perspective and environment.

Based on these criteria, Gardner initially presented the following seven Multiple

Intelligences: linguistic intelligence, musical intelligence, logical-mathematical intelligence, spatial intelligence, bodily-kinesthetic intelligence, and the personal intelligences (interpersonal and intrapersonal). Gardner added an eighth intelligence to his list in 1996, the naturalistic intelligence (Campbell, 1997), with a ninth, existential intelligence, added recently.

There are four key points to consider in MI Theory: "each person possesses all [nine] intelligences, most people can develop each intelligence to an adequate level of competency, intelligences usually work together in complex ways, and there are many ways to be intelligent within each category" (Armstrong, 1994, p. 11-12). With this in mind, a closer look at the multiple intelligences shows the expanse of knowledge these intelligences cover in the human sphere of thought, with particular attention to their application in mathematics:

Linguistic intelligence: "Sensitivity to the sounds, structure, meanings, and functions of words an language" (Armstrong 1994, p. 6). This intelligence type would be inclined toward writing down steps or discussing how to achieve a solution, with emphasis being placed on conveying mathematical meaning through the use of verbal or written communication.

Musical intelligence: "Ability to produce and appreciate rhythm, pitch, and timbre; appreciation of the forms of musical expressiveness" (Armstrong 1994, p. 6). Rhymes and chants can be used by students with this intelligence type to remember formulas like the quadratic equation, or to walk through how to solve for outliers in data sets.

Logical-mathematical intelligence: "Sensitivity to, and capacity to discern, logical or numerical patterns; ability to handle long chains of reasoning" (Armstrong 1994, p. 6). This is generally how mathematics is taught. Students generally can reason through answers, quickly see where ideas are going through inductive or deductive reasoning.

Spatial intelligence: "Capacity to perceive the visual-spatial world accurately and to perform transformations on one's initial perceptions" (Armstrong 1994, p. 6). Students with this intelligence type generally draw things out, and rely on diagrams and models to reason and convey mathematical meaning.

Bodily-kinesthetic intelligence: "Ability to control one's body movements and to handle objects skillfully" (Armstrong 1994, p. 6). This intelligence lends itself well to using mathematical instruments, such as rulers, compass, and protractor to create the movements required to solve and reason until a solution is found. Student may also turn their paper upside down, stand up to view something from a different angle, or use their bodies to consider ideas such as angle and length.

Interpersonal intelligence: "Capacity to discern and respond appropriately to the moods, temperaments, motivations, and desires of other people" (Armstrong 1994, p. 6). Students of this intelligence type will work well with groups to problem solve. They will ask questions to get the group thinking or to clarify misunderstandings. They may rely on the teacher and peers to help them reason or to find validation in their ideas.

Intrapersonal intelligence: "Access to one's own feeling life and the ability to discriminate among one's own emotions; knowledge of one's own strengths and weaknesses" (Armstrong 1994, p. 6). Students with strong intrapersonal intelligence may like working alone to problem solve, they might reread notes, look at examples, find possible contradictions and know where they are getting stuck to then search out individual understanding to come to an answer.

Naturalistic intelligence: "Expertise in distinguishing among members of a species; recognizing the existence of other neighboring species; and charting out the relations, formally, or

UCTM Recommended Book

By Mary Kay Stein, Margaret Schwan Smith

Five Practices for Orchestrating Productive Mathematics Discussions

In this book, we present and discuss a framework for orchestrating mathematically productive discussions that are rooted in student thinking. The framework identifies a set of instructional practices that will help teachers achieve high-demand learning objectives by using student work as the launching point for discussions in which important mathematical ideas are brought to the surface, contradictions are exposed, and understandings are developed or consolidated. The premise underlying the book is that the identification and use of a codified set of practices can make student-centered approaches to mathematics instruction accessible to and manageable for more teachers. By giving teachers a road map of things that they can do in advance and during whole-class discussions, these practices have the potential for helping teachers to more effectively orchestrate discussions that are responsive to both students and the discipline.

Throughout the book, we illustrate the practices in real classrooms with which we have become acquainted through research or professional practice (e.g., through teachers with whom we have worked in professional development initiatives). In particular, we make significant use of two classroom lessons: the Case of Darcy Dunn and the Case of Nick Bannister. The Case of Darcy Dunn is introduced in chapter 3 as a vehicle for investigating the five practices in action, and it is revisited in subsequent chapters as the practices are explored more fully. The Case of Nick Bannister is explored in considerable depth in chapters 4 and 5 as each of the five practices is examined in detail, and then it is referred to again in subsequent chapters as broader issues are considered.

Following research that has established the importance of learners' construction of their own knowledge (Bransford, Brown, and Cocking 2000), we have designed this book to encourage the active engagement of readers. In several places, we have provided notes (titled "Active Engagement") that suggest ways in which the reader can engage with specific artifacts of classroom practice (e.g., narrative cases of classroom instruction, transcripts of classroom interactions, instructional tasks, samples of student work). Rather than passively read the book from cover to cover, readers are encouraged to take our suggestions to heart and pause for a moment to grapple with the information in the ways suggested. By actively processing the information, readers' understandings will be deepened, as will their ability to access and use the knowledge flexibly in their own professional life. In addition, at the end of chapters 4, 5, 6, and 7, we have provided suggestions (titled "Try This!") regarding how a teacher can explore the ideas from the chapter in their own classrooms.

Although the primary focus of the book is the five practices model (chapters 1, 3, 4, and 5), it also explores other issues that support teachers' ability to orchestrate productive classroom discussions. Specifically, chapter 2 emphasizes the need to set clear goals for what students will learn as a result of instruction and to identify a mathematical task that is consistent with those learning goals prior to engaging in the five practices. Chapter 6 focuses explicitly on the types of questions that teachers can ask to challenge students' thinking and the moves that teachers can make to promote the participation of students in whole-class discussions. Chapter 7 situates the five practices model for facilitating a discussion within the broader context of preparing for a lesson and introduces a tool for comprehensive lesson planning in which the five practices are embedded. The book concludes with chapter 8, which discusses ways in which teachers can work with colleagues, coaches, and school leaders to ensure that they have the time, materials, and access to expertise that they need to learn to orchestrate productive discussions.

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informally, among several species" (Armstrong, 2009, p. 11). Students with naturalistic intelligence may see similarities in different mathematics concepts and ask questions to create the connections to group mathematics into similar idea families.

Existential intelligence: "the ability and proclivity to pose (and ponder) questions about life, death, and ultimate realities" (Howard Gardner's Multiple Intelligence Theory, n.d.). For students, this intelligence considers the broader applications of math to other areas and a consideration for essential questions that are posed at the beginning of a math unit.

The Multiple Intelligences in Teaching and Learning

Gardner expressed the value of his theory for education, stating that, "Only if we expand and reformulate our view of what counts as human intellect will we be able to devise more appropriate ways of assessing it and more effective ways of educating it" (2011, p. 4). If educators accept this multifaceted view of intelligence, we then must adopt methods for teaching to an audience of learners that express a variety of approaches to learning. There is a connection between a student's intelligence and how a student uses those intelligences to actually learn and achieve academically. More specifically, the multiple intelligences can be used to help students learn and achieve mathematically. The development of favored intelligences begins early. For instance, while a student may have ability in all nine intelligences, over time, certain intelligences are favored, or relied on more, whether through instinct, personal inclination, or cultural influence. Thomas Armstrong, a scholar of Multiple Intelligences, submits that, "[C]hildren seem to begin showing what Howard Gardner calls 'proclivities' (or inclinations) in specific intelligences from a very early age. By the time children begin school, they have probably established ways of learning that run more along the lines of some intelligences than others" (1994, p. 26). Therefore, it can be assumed that if children develop learning preferences geared toward certain favored intelligences during their early education, then, by the eighth grade, students have already begun solidifying their approaches to learning with regards to favored intelligences depending on their experiences both in and out of the classroom. Gardner relates, "Authorities generally agree that, outside of schooled settings, children acquire skills through observation and participation in the contexts in which these skills are customarily invoked. In contrast, in the standard classroom, teacher talk, often presenting material in abstract symbolic form and relying on inanimate media such as books and diagrams in order to convey information" (Gardner, 2011, p. 374). Seeing the stark contrast of out of class and in class learning as a result of the disparity between real world and in class teaching, the need for a balance between the two can be found in using the Multiple Intelligences.

In education, too often teachers resort to lecture because it is easier to do and there are far too many demands on teachers' time and resources. But to "be successful in educating all of our students, we need to be aware of their individual learning styles and multiple intelligences" (Snyder, 2000, p. 12). While it may seem daunting to try to teach to a classroom full of students that each have a unique set of learning styles and Multiple Intelligences, it is possible to create lesson plans that cater to different intelligences and allow for student expression of their intellectual profiles without having to create a different instructional model for each student in the class.

If educators embrace Gardner's theory as a framework for reaching the diverse learners in their classrooms, they must be prepared to act not only to include the theory into instruction, but also into assessment. The theory can be implemented into instruction by teaching a concept with access to each, or at least some, of the Multiple Intelligences, or by having students work on activities where the structure of the activities is framed around the intelligences that students will need to rely on to solve the problem/scenario. If we change the instructional format from lecture-based to that of a Multiple Intelligences approach, then assessment must reflect the instructional practice and bring in the Multiple Intelligences in its design and execution to achieve alignment and continuity for learners.

Applications in Math

Knowing the value of Multiple Intelligences as an approach to instruction and assessment, the question then becomes how do we implement the intelligences in a mathematics class and still meet the state and national standards as well as specific pedagogical goals that have been developed in an educator's teaching philosophy? Martin submits that, "If we wish to expand the mathematical horizons of our students, we must examine Gardner's theories and see how they can be incorporated into the teaching and learning of mathematics" (Martin, 1996, p. 4). This requires a teacher to reexamine the approach that is taken with regards to instruction and assessment to allow for the multiple intelligences to be built into and around the content, learning processes, and assessments that students experience to learn mathematics. Through the use of Multiple Intelligences, students experience the applicability to the real world as they work through problems using the intelligences they will use in other real world situations.

Martin asks, "How often have we heard in the math classroom, 'When are we ever going to use this?' This question is asked because, all too often, skills are isolated from their applications in the 'world' (Martin, 1996, p. 8). Through Multiple Intelligence, students can build skills based on their preferred intelligences, as well as gain experienced with their weaker intelligences, and they develop a sense of relevancy for mathematics in their lives. Unique problem solving using the Multiple Intelligences in a math class facilitates applying mathematics skills to real life because students are allowed to explore mathematics from various view points. They practice problem solving with different intelligences or experience the construction of mathematics through a Multiple Intelligences perspective and approach and this then translates to real life situation where students will need to be able to approach problems in a myriad of ways to create unique and notable solutions.

Assessment in mathematics is also an integral part of the learning process. In general, mathematics assessment relies heavily on multiple choice and paper-pencil testing to assess a student's aptitude in different skills and concepts. "Traditionally, assessment of students' learning in math has had a narrow focus and vision. The focus has been on paper-and-pencil tests; the vision has been to give the students a grade" (Martin, 1996, p. 10). Martin also stated that, "If we change what we teach in mathematics and the way we teach it, can we maintain traditional methods of assessment?" (1996, p. 9). The answer is no. We cannot expect to teach differently and test the same and arrive at improved results. If the aim is to use multiple intelligences to improve mathematics teaching and learning, then assessment must change to meet that demand and expectation. "Assessment of students should allow for their unique modes of learning and should enable them to present their knowledge in their own style. Assessment should provide opportunities for learning, it should be a beginning, not an end" (Martin, 1996, p. 10).

Through the use of Multiple Intelligences in the mathematics classroom students are able to discover the unique ways in which they approach learning because they begin to discover the intellectual capacities that they rely on in formulating ways in which to problem solve and therefore approach mathematics learning. Munro states that, "Helping student to understand and to value the uniqueness of their own approaches to learning is empowering; it gives them a base or starting point from which they can develop further" (Munro, 1994, p. 12).

Thomas Armstrong offered an example of how we can use the Multiple Intelligences within the instructional procedure. The goal was to teach students the role that x plays in an equation. Teaching through the intelligences could take the form of one, two, or more of these approaches to teach the same concept:

Linguistic: "Students are provided with a verbal description of x"

Logical Mathematical: "Students are given and equation and shown how to solve for x"

Spatial: "Students are told that x is like a masked outlaw that needs to be unmasked; student draw their own version of x"

Interpersonal/Bodily Kinesthetic: "Students act out an algebraic equation, where a student wearing a mask plays x, and the other students represent numbers or functions." In subsequent steps, one student moves the equation students around to get x by itself.

Bodily Kinesthetic: "students perform algebraic equations using manipulatives" or "students rhythmically repeat… lyrics several times" to learn a concept

Intrapersonal: "students are asked, 'what are the mysteries—or x's—in your own life?' discuss how students 'solve for x' in dealing with personal issues" (Armstrong, 1994, p 176-177).

Another approach is offered by Hope Martin. Martin follows with activities that are approached through Multiple Intelligences. For instance, she created an activity called Is this Game Fair? in which students must determine whether a game with a given rule is fair or not for both players. Outcomes and expected values are given. Students work to determine if the game is fair using knowledge from class, and using the following intelligences: logical/mathematical, interpersonal (students are working in pairs), intrapersonal, visual/spatial, and bodily/kinesthetic (students are rolling a pair of dice) (Martin, 1996, pg. 205-206).

With regard to assessment, mathematics educators must evaluate the utility of multiple choice testing. The majority of tests have very little relevance to real-world application of mathematical principles. They also deny students the opportunity to express their understanding of mathematics using their unique perspectives and intellectual approaches. The use of authentic assessments including projects, presentations, game-based approaches, written analyses, portfolios, and more might be better leveraged to more fully identify students' mathematical achievement.

Through the use of Multiple Intelligences as a way to approach teaching mathematics, more classrooms can become better suited to helping students develop skills for problem-solving and thinking mathematically through the use of intelligences they have.

Conclusion

With test scores showing little to no improvement on national and international levels, mathematics education cannot remain as is. Changes to both instructional design and assessment expectations must be realized to reengage students in mathematics and promote mathematical achievement. Through the theory of Multiple Intelligences, educators are able to tap into students' intellectual abilities to view problem solving as a multifaceted sphere and go beyond the logical/mathematical approach to teaching and learning. Teachers can embrace an instructional approach based in MI that brings new life and the real world of problem solving to the classroom.