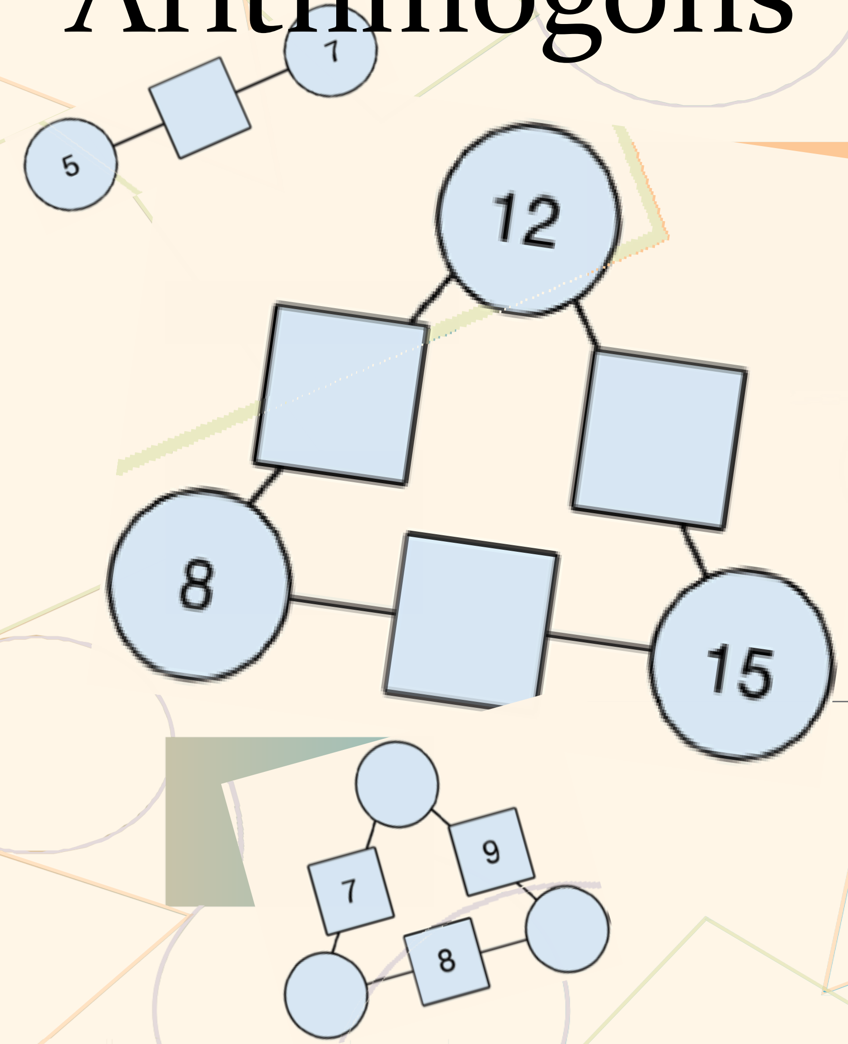
Utah Mathematics Teacher

Fall/Winter 2014-2015
Volume 7

Arithmogons



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Call for Articles

The ***Utah Mathematics Teacher*** seeks articles on issues of interest to mathematics educators, especially K-12 classroom teachers in Utah. All are encouraged to contribute articles and opinions for any section of the journal. Some of the features are: UCTM Leader Spotlight; Letter from the NCTM President; Letter from the UCTM President; Professional Development, Mathematics for English Language Learners; Puzzle Corner; Recommended Readings and Resources; Utah Core State Standards and Implementation; College and University Research; and others.

Teachers are especially encouraged to submit articles including inspirational stories, exemplary lessons, beginning teacher ideas; or managements tools. Sample ideas are (but not limited to) focused on teachers or districts who have successfully implemented the Utah Core, Inquiry based calculus, and new math programs K-12. Manuscripts, including tables and figures, should be typed in Microsoft Word and submitted electronically as an e-mail attachment to Christine Walker (Christine.Walker@uvu.edu). A cover letter containing author's name, address, affiliations, phone, e-mail address and the article's intended audience should be included. Items include, but are not limited to, NCTM affiliated group announcements, advertisements of upcoming professional meetings, and member updates.

Don Clark—Pam Jensen

I have been very fortunate during the past 43 years to combine my love of working with children and my love of math into a fun career and retirement. I spent 35 wonderful years working in Weber County School District including 19 years of teaching junior high math in Ogden Valley, 7 years supervising math and science at the district level and 9 years as an administrator "with a math agenda"! Since retiring in 2006, I have taught math to home school students in my home, tutored several pre-service teachers and helped several neighborhood students learn to enjoy math. I also enjoy the opportunity I have to teach courses for the Elementary Math Specialist program in Box Elder, Cache and Logan Districts. I'm very grateful for the wonderful co-workers I've associated with throughout the years and for the opportunity to attend UCTM and NCTM conferences to enhance my mathematics education skills.

Karl Jones—Julie Anzelmo

I graduated with a B.A. in Classical Languages and Literature from the University of Maryland at College Park in 1993, and completed an M. Ed in Teaching and Learning with an emphasis in literacy from the University of Utah in 2000. After spending a few years teaching adults in high-school-completion and university-level education courses, I decided to go where the real action is, and certified as an elementary teacher in 2004. I have been a teacher in the Salt Lake City School District ever since. I joined my district's Math Proficiency Network in 2008, which led me on the path to closely examining my own math teaching. I was urged by two colleagues, Julie Henderson and Marilyn Taft, to pursue National Board certification in Early Adolescence Mathematics that same year, and completed a Level 2 math endorsement shortly after achieving NB certification. After a brief sojourn into the world of academic coaching, I returned to the classroom, where I now teach 4th grade in Hawthorne School's Extended

Muffet Reeves—Stevane Godina

Stevane Godina has been a teacher for the Salt Lake City School District since 1982 and a math coach for the past twelve years. She enjoys designing and teaching professional development for teachers including Elementary Math Endorsement courses and has assisted in development of the math assessments for the Utah State Office of Education. The highlight of her professional career was in 2013-14 when Stevane had the opportunity to be a facilitator of the Math Professional Development Study for the American Institutes for Research and received instruction at the Harvard Graduate School of Education. She received her Bachelor's degree from University of Utah (Go UTES!) and her Master's degree from Southern Utah University. In addition to working with teachers, she loves traveling with her husband and spending time with her children and grandchildren.

Presidential Award Finalists

April Leder: I have been in education for 20 years. I am a former district math specialist, but I'm back in the classroom teaching third grade. I have two master's degrees and an Elementary Math Endorsement. I am an advocate for all children being taught to understand math concepts in a way that makes sense to them. Our students will have jobs that require skills such as communication skills, problem solving, and flexibility to solve a problem a different way if the first way doesn't work. We have to teach differently if we expect our students to be college and career ready.

Jalyn Kelley has been a professional educator for 14 years. She has completed Reading, ELL, Technology and Elementary Math endorsements. Through a State Technology Grant she created math podcasts aligned to the Utah 4th Grade Core Curriculum. She is a district math trainer and has presented at UCTM and NCTM. A year ago, Jalyn was invited to collaborate with Jessica Shumway, author of *Number Sense Routines*. Stenhouse Publishers videotaped her students as they worked through their number sense routines in their classroom. The video was released, Spring 2014. Check out a clip at: http://www.stenhouse.com/html/go-figure.htm?r=n315

Marsha Newman has begun her 6th year as a teacher at J. E. Cosgriff Memorial Catholic School, having taught both fifth and sixth grade. At Cosgriff, she also serves as the Math Coach, supporting teachers in their implementation of Singapore math strategies Pre-K to 6th grade to increase student learning. From 2003 - 2009 she was employed in Murray Public Schools as a fifth and sixth grade teacher. During the summer of 2009, Marsha was chosen to attend the Mickelson ExxonMobile Mathematics and Science Teachers Academy in Jersey City, NJ. She served as fifth grade mathematics presenter with the Utah Core Academy in the summers of 2007 and 2008. Marsha looks forward to helping students build an understanding of mathematics and providing them with tools to allow them to enjoy mathematics and become successful.

Renee Wakamatsu: I enjoy the challenge of having students enjoy math. For those who struggle as well as those who excel, I try to find ways to keep and build their confidence in their math abilities.
For the past four years, I have been fortunate to be a designated mentor for student teachers and cohorts from Brigham Young University and Utah Valley University. Math is the first subject for me to scaffold and hand over, as the method of real-world application and deeper understanding is difficult for a teacher to be prepared for.

I have served for many years as the Collaborative Team Leader (CTL) for the sixth grade team, focusing on developing our team and school as a Professional Learning Community (PLC). Currently, I am on the School Leadership team, which works with the principal in making administrative decisions that affect the entire school population.
I try to find experts from our community who become important resources to our school. One is a story of success for the school, students, and me.
Many times, at the end of the day, I take a breath to reflect on how much learning had taken place while we had so much fun through a busy day. Students internalize more of their learning if they experience what needs to be learned. By creating a connection to their own lives, the new information, or the new experience, will stay with them longer.

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Students' Can Take Us Off Guard
Jane M. Keiser, Ph.D., Miami University

What is The Utah Middle School Math Project?

The textbook materials were created in response to the Math Materials Access Improvement solicitation issued by the Utah State Office of Education in June of 2012. Hugo Rossi is the Principal Investigator for the work with Maggie Cummings and David Wiley serving as Co-Principal Investigators. The materials are a collaborative work with contributors from University of Utah, Utah State University, Snow College, and Weber State College; Jordan, Granite, Davis and Salt Lake City School districts; and many teachers throughout Utah. In particular, we would like to acknowledge the work of:

Mathematical Foundation Lead Writers:
Hugo Rossi
Jonathan Bodrero
Christine Walker

Workbook Lead Writers:
Maggie Cummings
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Allen Jacobsen
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Camilla Perkes Strong

Within the site http://utahmiddleschoolmath.org/ you will find textbook materials written specifically for the Utah Core Standards for both 7th and 8th grade. Included, you will find a foundational text for each chapter describing the mathematics students will be learning, a student workbook that has daily classroom activities and homework sets, and a teacher edition of the student workbook that has answers to problems sets and lesson suggestions. During the coming year, while these materials are being pilot tested, we will create supplementary materials, including videos, a "help-desk", glossaries, and an online adaptive assessment.

their calculations. Participants would need to notice that the place value of the recorded side values will always add one digit to the right each time as long as a zero was not in the decimal. This occurred in the answer of

$$41.\overline{230769}.$$

This rich discussion also assumes that the student finds the maximum amount to subtract each time, as done with long division.

If the classroom discussion was successful in placing parameters on a Seth-like approach, it would increase his confidence and help him to be more efficient in the use of his method. The rest of the class would benefit by seeing relationships between their own approach and Seth's; there would be a greater understanding of the operation of division in general; and creative individuals, such as Seth, would know that their thinking was valued in the classroom.

At the same time, students might recognize the beauty of not having to think so carefully about place value at each step, which is nicely provided when we use the long-division algorithm. Students will not move to a more efficient method if they never realize that a better way is possible. In working with independent thinkers, we decided that the side-by-side modeling of students' methods could be a good way to elicit these epiphanies.

CONCLUSION

Our group of middle-grades teachers had decided that having in-class discussions on the basis of students' correct and incorrect strategies was a better use of class time, since these teachers were very committed to fully implementing the CMP curriculum. We followed 84 students from the fall 2008 sixth grade to the fall 2010 eighth grade. We found that these small attempts to integrate side-by-side comparisons of student work into the normal curriculum have helped them in their computational abilities, even though these abilities were clearly not the focus of the CMP curriculum.

The sixth-grade students had a 30 percent accuracy rate on the 965 + 16 problem; this same set of eighth graders had a 54 percent accuracy rate on the 765 + 15 problem. More important, 40 percent of this group of students shifted from a less-efficient strategy to a more-efficient approach. For example, a student shifted from using clusters of multiplication and adding up to the dividend to using repeated subtraction effectively or from repeated subtraction to the long-division algorithm. Table 1 shows three of these pre-strategy and poststrategy shifts from this class of students.

When students progress to the middle grades with a conceptual grounding rather than with a skills-based background, it is up to us, their teachers, to use their strengths to help them with their weaknesses. Since our students are becoming good and confident problem solvers who will invent strategies if they do not know a method, we should promote that confidence by sharing the correct and incorrect thinking with all our students. We should also help them by modeling good ways of testing our inventions to make sure they are mathematically sound.

Our students are no longer open to memorizing an algorithm without understanding it. Therefore, attempting to make all students use the long-division algorithm is a practice that we have decided to eliminate. Instead, we hope that students will learn to value the efficiency of such an approach when it is modeled side by side with other student approaches. Fuson (2003) recommends the following regarding students' computational fluency:

> Clearly, the twenty-first century requires a greater focus on a wider range of problem-solving experiences and a reduced focus on learning and practicing by rote a large body of standard calculation methods. How to use the scarce hours of mathematics learning time in schools is a central issue. (p. 301)

Our hope is that by honoring our students' creative abilities, we will keep them involved and still open to improving their computational skills.

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Keiser, Jane M. 2010. "Shifting Our Computational Focus." *Mathematics Teaching in the Middle School* 16 (4): 216–23.

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Jane M. Keiser, keiserjm@muohio.edu, teaches preservice mathematics teachers at Miami University in Oxford, Ohio. She is interested in students' conceptual understanding of the four mathematical operations and has enjoyed her partnership with her colleagues in the local Talawanda School District. She thanks Karen Fitch, Don Gloeckner, Brad Engel, Megan Murray, and Bob George for their help with this article.

Presidents Message

Travis Lemon, UCTM President

It is time to take Action! With the release of "Principles to Action" by the National Council of Teachers of Mathematics (NCTM, 2014) this past April, we as a profession of mathematics teachers are once again provided with great vision and direction by our professional organization. NCTM is leading the charge and setting the pace when it comes to an agenda for change, profession, and improvement.

Within the *Principles to Actions* document NCTM described eight "Effective Teaching Practices" that are high-leverage, research-based and results oriented. They are not just eight nice 'to do' items, but rather are founded in a significant body of research, experience and practicality. It is a significant and important message about teaching and learning mathematics that we should use as a means for self-reflection, goal setting and improvement of our practice.

I would encourage all of you to take the opportunity to obtain a copy, read it and use it as a means to reflect on and improve your practice. For those of you that may not have had the opportunity to see the eight Effective Teaching Practices, I have listed them below and for all of us, I think it is worth reading and reflecting on these items and looking into them at a deeper level. Every time I read over this list, a different teaching practice resonates more with me, and causes me to think about what I am doing as a teacher to produce great learning experiences.

Mathematics Teaching Practices

- Establish mathematics goals to focus learning.
- Implement tasks that promote reasoning and problem solving.
- Use and connect mathematical discourse.
- Pose purposeful questions.
- Build procedural fluency from conceptual understanding.
- Support productive struggle in learning mathematics.
- Elicit and use evidence of student thinking.

On the heals of the new core standards and newly created end of level testing, I am thinking more intently about conceptual understanding as a foundation

Presidents Message Continued

on which procedural fluency can be built. What does it mean to promote depth of mathematical knowledge? How can I better understand the mathematical concepts and use or create tasks that will get at the heart of the mathematics for students? What implications does this teaching practice have for me and my students? I know that mnemonics and other memorization or memory devices can be of benefit, however the introduction to the core standards stated the following with respect to mathematical understanding.

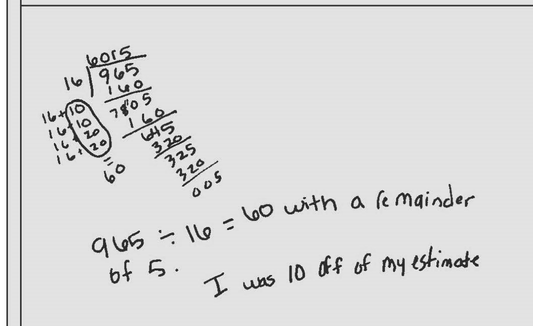
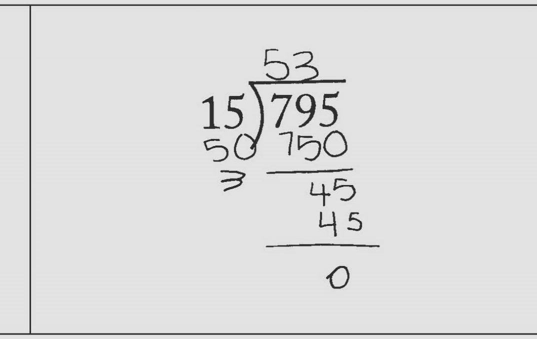
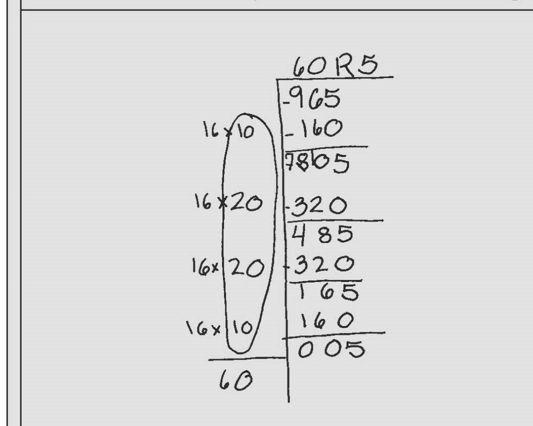
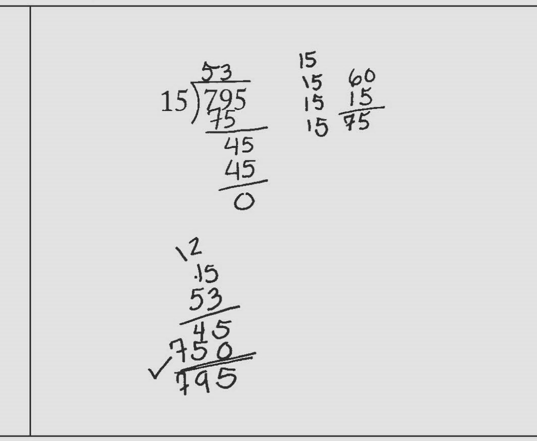
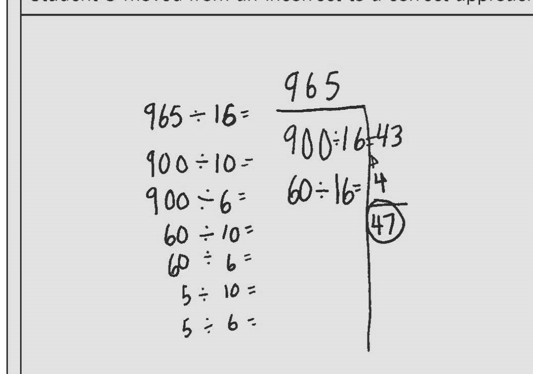
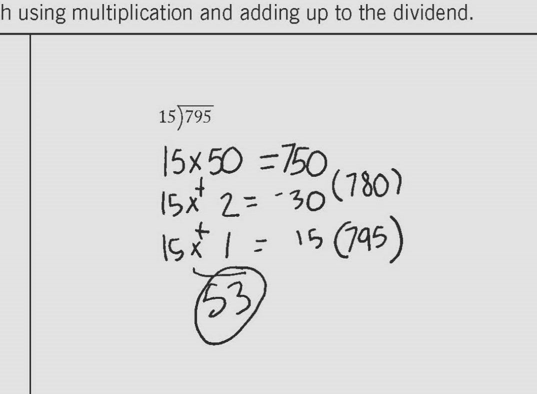
> "But what does mathematical understanding look like? One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student's mathematical maturity, why a particular mathematical statement is true or where a mathematical rule comes from. There is a world of difference between a student who can summon a mnemonic device to expand a product such as (a+b)(x+y) and a student who can explain where the mnemonic comes from. The student who can explain the rule understands the mathematics, and may have a better chance to succeed at a less familiar task such as expanding (a+b+c)(x+y). Mathematical understanding and procedural skill are equally important, and both are assessable using mathematical tasks of sufficient richness."

I think in place of mnemonics I might consider a task that promotes reasoning and problem solving where students can discuss, pose purposeful questions, and use as well as connect mathematical representations. This may take more time, effort and willingness on my part. However, the depth of knowledge and learning for understanding that can result is worth the effort.

For me, and I am sure for us all, the Effective Teaching Practices present a challenge as well as a chance for growth and progress. As you read the document more closely you will see that a reflection on productive and unproductive beliefs is provided. It is my hope that I can become increasingly more productive and that my students will be the beneficiaries of my efforts. May we all engage more intently in realizing more the vision of more and better mathematics for all our students.

Travis Lemon
President, UCTM
2013—2014

Table 1 Three students shifted strategies when progressing from the sixth grade to the eighth grade.

Sixth Graders Solving 965 ÷ 16	Eighth Graders Solving 795 ÷ 15
Student 1 moved from repeated subtraction with many subtractions to repeated subtraction with very few subtractions.	
	
Student 2 moved from repeated subtraction to the long division algorithm with a check of the multiplication.	
	
Student 3 moved from an incorrect to a correct approach using multiplication and adding up to the dividend.	
	

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However:

$$4=\frac{20}{5}=\frac{1+1+8+10}{5}=\frac{1}{5}+\frac{1}{5}+\frac{8}{5}+\frac{10}{5}=4$$

From these two examples, breaking up the numerator and not the denominator will work correctly, but rewriting the denominator as a sum will not work. In fact, had the two students in **figure 3** written 965 ÷ 16 as

$$\frac{900+60+5}{16}=\frac{900}{16}+\frac{60}{16}+\frac{5}{16}=56\frac{1}{4}+3\frac{3}{4}+\frac{5}{16}=60\frac{5}{16},$$

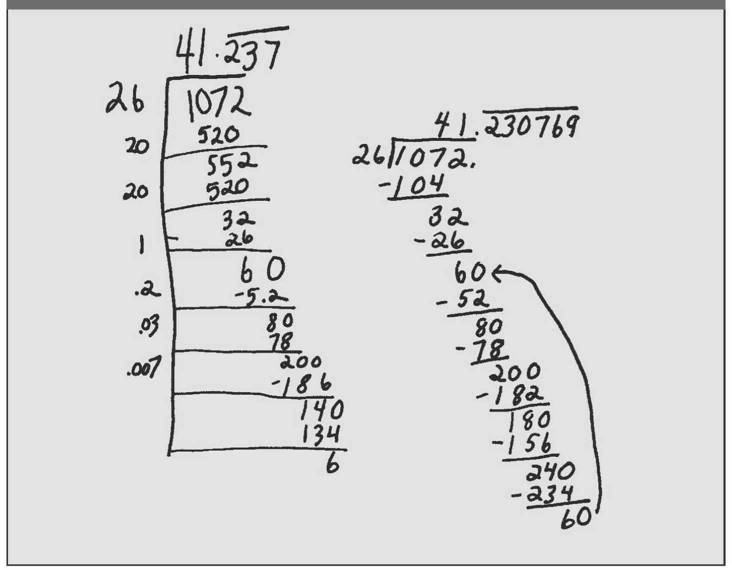
they would have produced the correct answer. It is not a convenient or efficient approach, but it works. See, for example, rewriting the numerator as a sum of convenient numbers:

$$\frac{320}{16}+\frac{640}{16}+\frac{5}{16}=20+40+\frac{5}{16}=60\frac{5}{16}$$

The division resembles other successful strategies shared earlier in the article.

Allowing students to analyze others' methods requires a trusting and comfortable problem-solving environment. However, the conversations alone help students learn how to test their ideas for themselves with simpler cases so that they will not simply apply mathematical properties to situations that will not work. These conversations help students reflect on the conceptual and notational features of each strategy or algorithm, as well (NRC 2001).

Fig. 5 Seth's work (at left) was modeled (at right) with long division.



A CORRECT APPROACH FOR 1072 ÷ 26

The approach used on the left in **figure 5** was "invented" by an eighth grader we will call Seth. When discussing it with the teachers, we found that a sixth-grade student had produced the same idea when her class was exploring division past the decimal point. Both students must have felt very comfortable with the concept of "division as repeated subtractions of groups of the divisor."

When Seth got to the "remainder" of 6, he added a 0, but he might have been thinking of it as a 6 when he recorded the 0.2 at left. Rather than thinking, "How many 26s can I subtract from 6?" he may have thought, "What fraction of 26 can I take away from 6?" If he took 1/10 of 26, that would be only 2.6, but 2/10, or 0.2, would be 5.2, which would be much closer. This student did remarkably well until he reached this point: "What fraction of 26 can I take away from 0.020?" Because he could not take 1/1000 of 26 away, which would be 0.026, he brought down a second 0 so that he could take 0.0007 26s away from 0.0200. However, he recorded the place value incorrectly as 0.007 *and* multiplied 7 × 26 and found 186 instead of 182.

When I questioned a sixth grader who had used Seth's approach about the difficulty of keeping the place values straight, she responded that she understood it better this way, so why change? Although my own mental abilities do not readily adjust to taking *x* tenths, hundredths, or thousandths of a number, I can argue that this student *did* understand her mathematics. In fact, these two students' invented algorithm has meaning attached to the set of steps, whereas long division loses meaning when it fails to acknowledge place value.

Having a student or teacher model the long division algorithm next to Seth's work could produce a rich conversation. It could also include how Seth's method could be made to work every time so that students using it could be more accurate and efficient in

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Moving Ahead: Opportunities & Priorities

by NCTM President Diane J. Briars
NCTM *Summing Up*, May 6, 2014

As I begin my term as NCTM President, I'm struck by both the challenges currently facing the mathematics education community and the opportunities for systemic improvement in mathematics teaching and learning that addressing these challenges affords. One of our major challenges is, of course, addressing the Common Core State Standards for Mathematics (CCSSM)—facilitating large-scale, effective implementation; preparing for more rigorous, aligned assessments in spring 2015; and supporting the standards themselves.

Although preparing teachers and administrators across the country to implement CCSSM is an enormous challenge, it is also an unprecedented opportunity to widely disseminate features of high-quality mathematics programs that will effectively implement CCSSM and other college- and career-readiness standards. Principles to Actions: Ensuring Mathematical Success for All, NCTM's new signature publication released at the 2014 Annual Meeting, does just that. It presents six Guiding Principles for School Mathematics that are essential for high-quality mathematics programs, along with eight research-informed Mathematics Teaching Practices that help students develop the conceptual understanding, problem solving, reasoning, and procedural fluency called for by CCSSM and other high-quality standards. Principles to Actions builds on the Council's previous standards publications and concisely summarizes the features of effective mathematics instruction from research and experience. At the same time, it provides specific descriptions and examples of what these features look like in practice, the conditions needed to support their implementation in all classrooms, and recommended actions for teachers, school-based and district leaders, and policymakers to put these practices in place. This new publication will drive the Council's efforts to ensure equitable mathematics learning of the highest quality for all students.

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I encourage all of you first to read *Principles to Actions* and then to act to improve mathematics teaching and learning in your setting. In this new publication, NCTM raises key questions for all stakeholders to consider:

- For teachers:

◊ To what extent are your instructional practices consistent with the Mathematical Teaching Practices?

◊ Do your students have regular opportunities to engage in tasks that involve reasoning and problem solving?

◊ How do you support your students when they struggle with a task?

◊ To what extent do your assessments provide useful and timely information about students' mathematical knowledge?

◊ How are you and your students using the results to increase learning?

◊ What supports do you need to fully implement the Mathematical Teaching Practices?

- For school-based leaders:

◊ To what extent are all your teachers implementing the Mathematical Teaching Practices?

◊ What supports will they need to do so?

◊ Do your school's policies and practices promote or hinder teachers' implementation of these practices?

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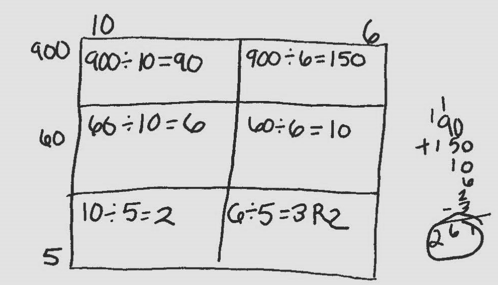
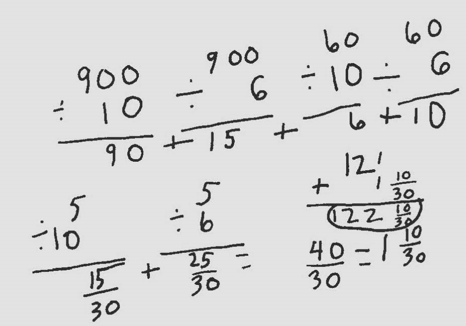
Fig. 3 Incorrect solutions by sixth graders when evaluating 965 ÷ 16 revealed much about their past math experiences and understanding of the distributive property.

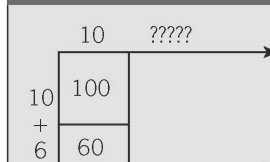


(a) Array model

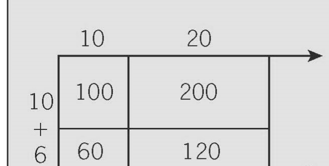


(b) Division algorithm

Fig. 4 One possible way to use the array model "backward" allowed the missing factor to be found.



The student asks, "What does the length need to be so that 965 is the area of the rectangle?" If it's only 10, I've only used up 160 of the 965 and still need 805 more.



The student notices that even with a length of 30, the area is only 480. Doubling the length (60) would create an area of 960. The student then tries a length of 61 and sees that the area would be 976, which is too much, so the answer must be 60 r5.

one side of the rectangle when the area was 965 and the other side was 10 + 6, she could have been quite successful. **Figure 4** shows a series of steps that would have helped the student use the array model to find the missing factor (division), rather than the product (multiplication).

Both examples in **figure 3** contained misconceptions:

$$965 \div 16 = \frac{900+60+5}{10+6} = \frac{900}{10}+\frac{900}{6}+\frac{60}{10}+\frac{60}{6}+\frac{5}{10}+\frac{5}{6}$$

The second student's work—had the approach been correct—actually showed some promise because the fraction addition was accurate. Many of our students produced similar work, which begged the question: "Why is this wrong?"

A seventh-grade teacher and I discussed the many instructional routes we could take. The worst approach would be to tell the class that the distributive property is only for multiplication and that it does not work for division. We thought that students would continue to wonder *why* it does not work in the same way. Instead, we returned the incorrect solution to the entire class and ask them to analyze the method for themselves. Does it give the correct solution? If not, does it ever work to rewrite the numerator *and* the denominator of the division fraction as a sum and then split it into separate fractions?

We hoped that students would discover situations that do not work as well, such as:

$$4 = \frac{20}{5} = \frac{20}{1+1+1+1+1} \neq \frac{20}{1}+\frac{20}{1}+\frac{20}{1}+\frac{20}{1}+\frac{20}{1} = 100$$

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percent (27 of 91) found a reasonably correct answer, after giving responses of 60, 61, and 60 r5. Some students answered the question and rounded up to 61; others just stopped with an answer to the computation.

Although we were discouraged by our students' lack of proficiency, we also found evidence in our preassessments that our students had more conceptual understanding than past students. Many used interesting strategies and exhibited much creativity, and we wanted to draw on these strengths to improve their computational fluency. Because the large majority of our *Connected Mathematics Project* (CMP) curriculum does not focus on whole-number computation, we needed to consider the amount of time that we would spend helping our students improve their computational skills.

Some of our middle-grades teachers had been teaching the long division algorithm to a majority of students who either said they had never seen it or who preferred to use a different strategy. Our preassessment revealed that few of our sixth-grade students used the long-division algorithm correctly and efficiently, so we decided that direct instruction of this algorithm would not be the best approach.

> ***When students progress to the middle grades with a conceptual grounding rather than with a skills-based background, it is up to us, their teachers, to use their strengths to help them with their weaknesses.***

The students were willing to engage in problem solving and sense making, and we decided that this was a strength to build on. We asked them to examine samples of division computation, both correct and incorrect, to help them think about the mathematical soundness of their own strategies.

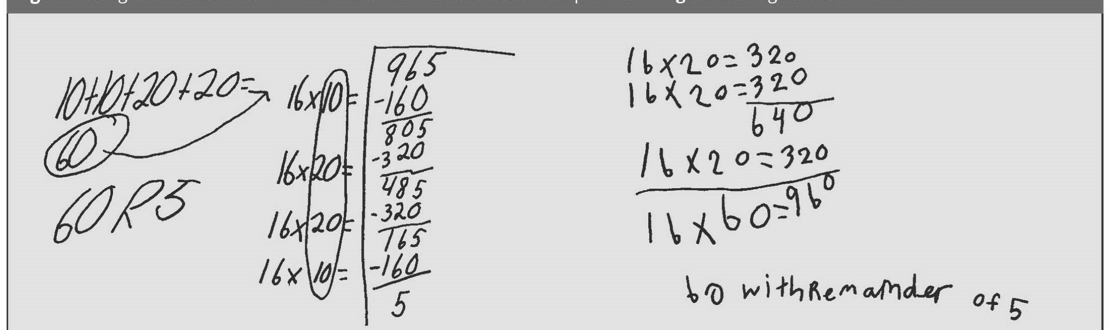
The teachers agreed to be on the lookout for instructional moments that occurred within the CMP curriculum to have students analyze others' methods and compare their ideas. Perhaps observing others' efficiency in using the long-division algorithm or more efficient approaches would encourage students to make a shift in their use of strategies.

It was hoped that discussions on correct and incorrect approaches could benefit students and help them learn strategies and be able to either discredit or confirm an approach. Two approaches to solving division problems follow, having been drawn from the students' preassessment. It is hoped that these approaches will provide fruitful discussions.

AN INCORRECT APPROACH FOR 965 ÷ 16

The student work in **figure 3** shows the same misapplication of the distributive property to the division problem 965 ÷ 16. In **figure 3a**, the student incorrectly used the array (or area) model that was normally reserved for multiplication, with the placement of both factors along the sides and the product in the center. Had she tried to use the array backward, to find the length of

Fig. 2 Sixth-grade students solved 965 ÷ 16 from the contextualized problem in **figure 1** using different methods.



(a) Using repeated subtraction

(b) Summing several groups of 16 to get as close to 965 as possible

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- For district or state leaders or policymakers:

◊ To what extent are teachers implementing the Mathematical Teaching Practices?

◊ Are school-based leaders prepared to support teachers in this implementation?

◊ What supports do teachers and leaders need to do so?

◊ Are your systems' practices and policies—for example, curriculum, assessments, and professional learning experiences—consistent with the Guiding Principles?

Principles to Actions is a powerful tool for advocating for effective teaching practices and the supports needed to implement them, as well as a detailed guide for individual and collective study and reflection. I strongly encourage you to use it in both ways. In short, read it, share it, act on it. Additional information about *Principles to Actions* includes an Executive Summary, webcasts of the 2014 Annual Meeting *Principles to Actions* sessions, and a Reflection Guide.

The Council will continue its active support of, and advocacy for, the Common Core State Standards for Mathematics. As stated in its August 2013 Position Statement, NCTM believes that CCSSM offers "a foundation for the development of more rigorous, focused, and coherent mathematics curricula, instruction, and assessments that promote conceptual understanding and reasoning as well as skill fluency." In particular, the focus and coherence of the standards in grades K–8—addressing fewer different topics in each grade, with careful attention to the progression of topics across grades—ensure the instructional time needed to implement the research-informed Mathematics Teaching Practices described in Principles to Actions. These are not new practices but are ones that the Council has long supported, including engaging students in solving tasks that promote problem solving and reasoning, followed by productive discussions about their work. Also, the continuing emphasis on conceptual understanding, problem solving, and reasoning in the high school standards, along with explicit attention to mathematical modeling will better prepare students for post-secondary education and/or careers. NCTM's reasoning and sense making initiative called for such emphases in high school mathematics as

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well. We will also continue to support the implementation of assessments that measure students' proficiency in all aspects of CCSSM expectations, as well as research related to CCSSM and its implementation that will inform future refinements.

At NCTM we are undertaking two initiatives to enhance our service to members and, more broadly, to further our mission of providing professional learning in support of equitable mathematics learning of the highest quality for all students.

First, we are engaging in strategic planning related to all of our professional learning opportunities. Our goal is to examine the wide range of options available, including face-to-face meetings, conferences, and online courses, to develop a suite of offerings that will provide effective professional learning experiences for all members of the mathematics education community. The first step in this process has been to ask the question,

What is the optimal time of year to hold our flagship professional learning event, the Annual Meeting and Exposition, to maximize the number of teachers who would be able to attend and enable attendees to best use information they learn at the conference?

Our answer took into account input from a variety of stakeholders over the past 18 months: Hold the Annual Meeting in the fall beginning in 2020, with regional conferences and other professional learning opportunities then scheduled accordingly. This strategic planning effort will continue over the next few months.

Second, to increase service to members and the community, the Council will launch a new website this fall. The new website will feature enhanced content and improved site navigation, along with other new features, and the site will be fully accessible on mobile devices.

Supporting high-quality early childhood education and strengthening the pathways from high school to college mathematics are two additional priorities for my work over the next two years. These topics are particularly important and timely, owing to recent research and developments in these areas.

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Procedural Instruction Doesn't Always Lead to Good Performance

The low proportion of procedural fluency among middle school students is not new to U.S. mathematics educators. A division problem on the 1996 eighth-grade National Assessment of Educational Progress (NAEP) showed that only 35% of students across the nation could correctly answer the question below, and calculators were available. However, research does not support a return to the other extreme in which instruction simply drills students on procedures.

> Anita is making bags of treats for her sister's birthday party. She divides 65 pieces of candy equally among 15 bags so that each bag contains as many pieces as possible. How many pieces will she have left?
>
> a. 33
> b. 5
> c. 4
> d. 3
> e. 0.33

Kaasila, Pehkonen, and Hellinen (2010) studied Finnish preservice teachers and high school students whose computational education consisted of an early presentation of the standard algorithms from second grade on. They reported in 2010 that only 30 percent of each group could correctly find 491 ÷ 6 if told that 498 ÷ 6 = 83 and told *not* to use the standard long division algorithm. Years of procedural instruction and relative success in the school system failed to provide these students with the conceptual understanding necessary to solve such a nonroutine problem.

These changes have produced both benefits and challenges. Students are thinking independently and willing to be creative in all areas of problem solving. Middle-grades students are also much better at mental math than they used to be. However, from a pedagogical perspective, following these same students' thinking has become more difficult because they have learned a variety of strategies for each operation. Because some students continue to use inefficient, cumbersome, and time-consuming strategies, teachers struggle to get them to accept more efficient mathematics.

This article shares examples of division computations completed early in the school year by sixth, seventh, and eighth graders. As a university partner, I was able to lead all but one of our local middle-grades mathematics teachers in a professional development workshop that focused solely on whole-number computation. We began the workshop by asking students in all three grade levels to take the same short assessments of multiplication (see Keiser 2010) and division so that we could get an idea of their level of proficiency. The teachers and I used this student work to plan the teachers' future computational interventions in the classroom for that academic year. An early inspection of students' work gave us much insight concerning ways in which these students' K–5 preparation had changed.

The division assessments given to the students in the sixth through eighth grades consisted of two problems. One was a "naked" problem, meaning *without context*, in which the division symbol was already written in the problem; the other was a story problem involving division (see **fig. 1**). Both problems asked first, for an estimate, and second, for a solution. In analyzing these preassessments, we saw primarily two division strategies:

1. Repeated subtraction of the divisor in groups (see **fig. 2a**)
2. Using multiplication and adding up to the dividend (see **fig. 2b**)

Very few students used the long-division algorithm. In fact, out of one teacher's 91 sixth-grade students, only 4 used the standard long-division algorithm, and only 2 used it correctly.

When this same group of students was asked to solve 965 ÷ 16, only 30

Fig. 1 Despite the fact that both these problems involved division, the estimates and approaches that students used varied.

A "NAKED" DIVISION PROBLEM
What is 1072 ÷ 26? First estimate and show how you found your estimate, then solve.

A CONTEXTUAL DIVISION PROBLEM
The FFA group at Greendale High School just had its fruit sale. The group receives a shipment of 965 navel oranges in several large crates. Students need to repackage the oranges in smaller boxes that will hold 16 oranges each. How many boxes will the FFA group need to be able to package all the oranges?

1. Give an estimate. Show how you found your estimate.
2. Solve.

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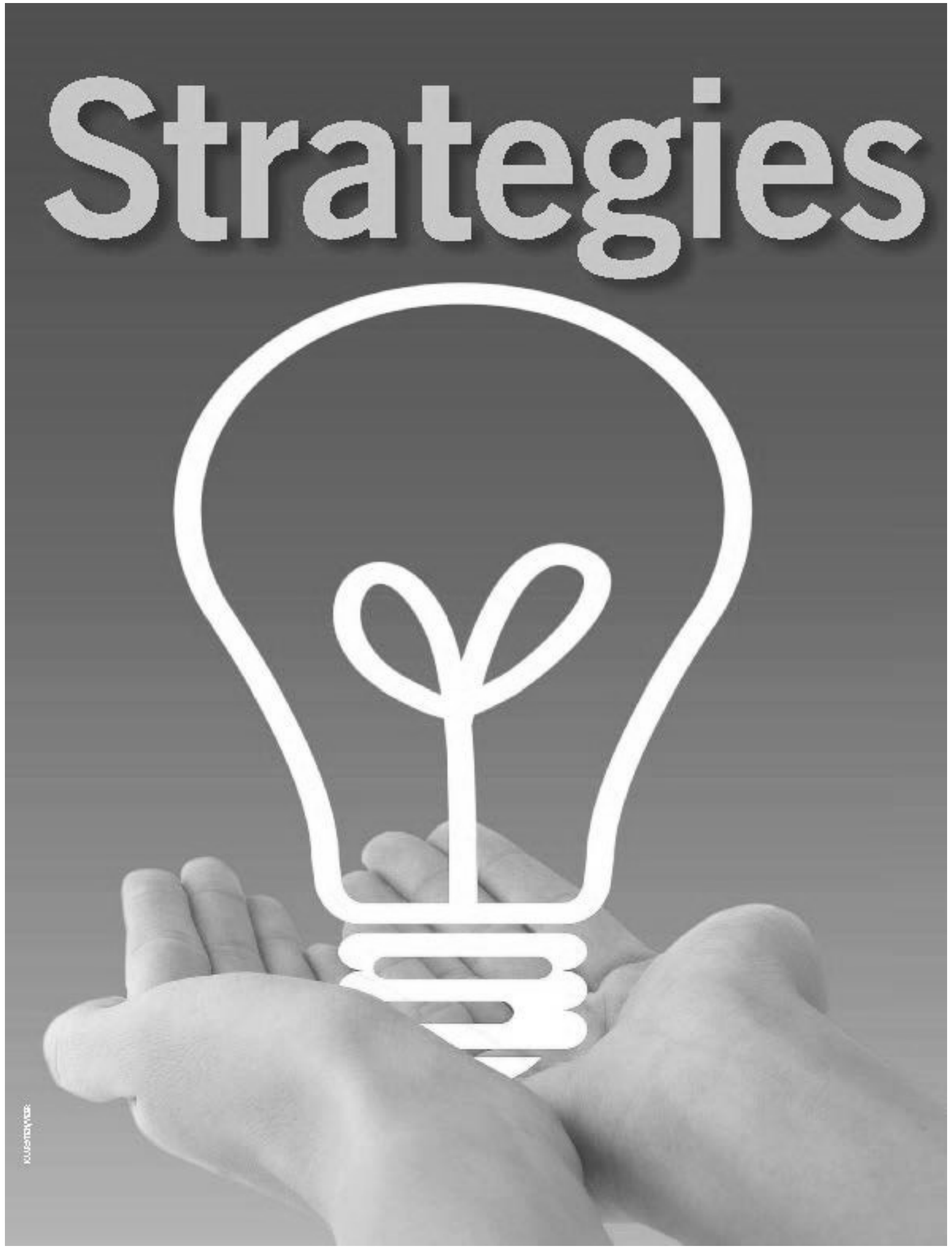

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The foundation for the early childhood priority is the NCTM Mathematics in Early Childhood Learning Position Statement, approved by the Board in October 2013, which calls for young children in every setting to have the opportunity to experience mathematics through effective, research-based curricula and teaching practices. Every child needs access to high-quality preschool and full-day kindergarten programs to develop understanding of early mathematics concepts.

A number of recent reports, including *Mathematics in 2025* (2013) and *What Does It Really Mean to Be College and Career Ready?* (2013), released by the National Research Council and the National Center on Education and Economy, respectively, highlight the increasing importance of statistics, modeling, and discrete mathematics in today's society, and the need to update our current curriculum pathways from high school to post-secondary education to prepare students mathematically for their futures. Consequently, this is the ideal time for NCTM to collaborate with other members of the Conference Board for the Mathematical Sciences, including the Mathematics Association of America, the American Mathematical Society, the American Statistical Association, the Society for Industrial and Applied Mathematics, and the American Mathematical Association of Two Year Colleges, as well as mathematics education organizations such as the Association of Mathematics Teacher Educators, the Association of State Supervisors of Mathematics, and the National Council of Supervisors of Mathematics to examine new alternatives.

I'll be writing more about each of these priority areas in future columns.

I am extremely honored to have the opportunity to serve as NCTM President and am very much looking forward to working with and for all of you over the next two years. NCTM is your organization. Please get involved and help NCTM become an even stronger support for you as a mathematics teacher, leader, teacher educator, or researcher.

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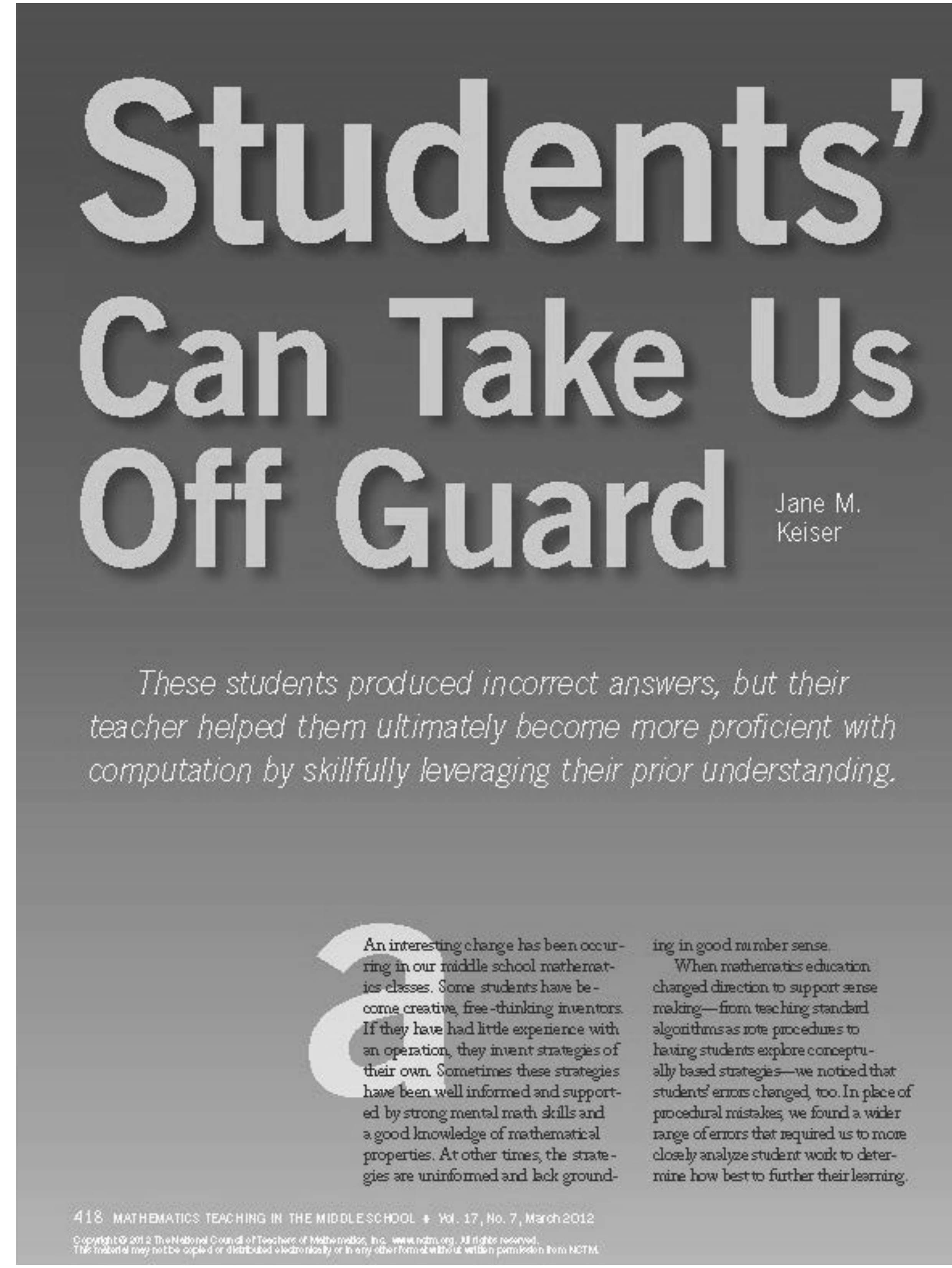
I Can Draw That! Effective Problem Solving Instruction Using Bar Models

Diane Halbasch— Davis School District
Kristin Hadley—Weber State University

Abstract

The goal of mathematics instruction is to help students apply their mathematical knowledge to problem solving situations. The purpose of this study was to compare the problem solving abilities of two groups of 6th grade students. The first group was taught to use bar models in a combination approach and the second group was taught to use problem solving steps and strategies in a combination approach. Four 6th grade classes from a suburban elementary school in northern Utah were taught to apply their knowledge of ratios to problem solving situations over a 3 week period. Gains in problem solving ability were measured using a pretest before the treatment and a posttest at the end of the treatment. Confidence levels were measured by a six question student survey. Data showed that students who were taught using bar models made statistically significant gains over students who were taught using problem solving steps and strategies. While no statistically significant difference was found, students across all proficiency levels and groups made gains and felt confident in their problems solving abilities. Teachers should seek training in using bar models as another effective means to help students become successful problem solvers.



Students' Can Take Us Off Guard

Jane M. Keiser

These students produced incorrect answers, but their teacher helped them ultimately become more proficient with computation by skillfully leveraging their prior understanding.

An interesting change has been occurring in our middle school mathematics classes. Some students have become creative, free-thinking inventors. If they have had little experience with an operation, they invent strategies of their own. Sometimes these strategies have been well informed and supported by strong mental math skills and a good knowledge of mathematical properties. At other times, the strategies are uninformed and lack grounding in good number sense.

When mathematics education changed direction to support sense making—from teaching standard algorithms as rote procedures to having students explore conceptually based strategies—we noticed that students' errors changed, too. In place of procedural mistakes, we found a wider range of errors that required us to more closely analyze student work to determine how best to further their learning.

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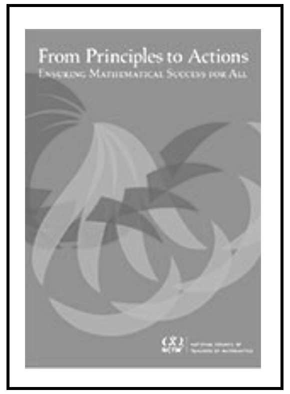
UCTM Recommended Book



By Steven Leinwand, American Institutes for Re search [and 8 others]

Principles to Actions

The widespread adoption of college- and career-readiness standards, including the Common Core State Standards for Mathematics, presents a historic opportunity to improve mathematics education.

What will it take to turn this opportunity into reality in every classroom, school, and district?

Continuing its tradition of mathematics education leadership, NCTM has defined and described the principles and actions, including specific teaching practices, that are essential for a high-quality mathematics education for all students.

Principles to Actions: Ensuring Mathematical Success for All offers guidance to teachers, specialists, coaches, administrators, policymakers, and parents:

- Builds on the Principles articulated in Principles and Standards for School Mathematics to present six updated Guiding Principles for School Mathematics
- Supports the first Guiding Principle, Teaching and Learning, with eight essential, research-based Mathematics Teaching Practices
- Details the five remaining Principles—the Essential Elements that support Teaching and Learning as embodied in the Mathematics Teaching Practices
- Identifies obstacles and unproductive and productive beliefs that all stakeholders must recognize, as well as the teacher and student actions that characterize effective teaching and learning aligned with the Mathematics Teaching Practices

With *Principles to Actions*, NCTM takes the next step in shaping the development of high-quality standards throughout the United States, Canada, and worldwide.

Mathematical instruction consists of learning concepts and skills that are numeric, algebraic, geometric, and statistical. These concepts and skills are of no use to students unless they are taught to apply them in problem solving situations. The National Council of Teachers of Mathematics (NTCM) stated,

> Each school district must develop a complete and coherent mathematics curriculum that focuses, at every grade level, on the development of numerical, algebraic, geometric, and statistical concepts and skills that enable all students to formulate, analyze, and solve problems proficiently. (NCTM, 2000, p. 74)

Unfortunately, students in the United States rank low when compared to students from other countries in their ability to apply their mathematical knowledge in problem solving situations. In 2009, The Program for the International Student Assessment (PISA) reported the United States ranked 25th out of 30 included countries in mathematical problem solving (NCES, 2009).

Clearly, teachers across the United States must improve mathematical problem solving instruction or students will continue to lag behind other countries and fail to apply mathematical knowledge appropriately. Problem solving is a complex process that integrates many different areas (Goldin, 2000). Instructional strategies have been developed that target areas of difficulty and help students improve their problem solving abilities.

The bar model approach is a relatively new strategy in problem solving in the U.S. This method is used in Singapore, among other countries, for math problem solving. With this strategy, students are taught to use concrete objects to make sense of mathematical problems.

As student thinking progresses, they are taught to model their concrete representations using a drawing of rectangular bars. These bar models help students visualize the structure of the problem. The method helps students see the relationship of given quantities within the word problem and enhances their thinking and problem solving skills. This strategy can be used with simple to complex problems that may require many steps. The method can be used with problems that use all four mathematical operations as well as problems that require the use of ratios, fractions, and percents. Hogan and Forsten (2007) stated that the bar model approach can be used in roughly 80% of the problems presented in math textbooks at the elementary school level. This method also serves as a good foundation for algebra. Kho (1987) stated, “students’ experience in using bars to represent quantities in the Model Method would enable them to appreciate better the use of letter symbols to represent quantities when they later learn the algebraic method” (p. 34).

Forsten (2010) has taken the bar model method and incorporated it into a combination approach to problem solving. In this approach, students use an explicit, seven step process that guides them through the problem solving process and helps them construct a bar model for the problem. After explicitly teaching students these steps, teachers act as facilitators as students use the steps to construct a bar model that correctly shows the relationships and structure of the problem. During this process, students use elements of a constructivist approach as they collaborate with each other in deciding who and what is involved in the problem, how to draw the bar model representation, and working the computation necessary to solve the problem. After solving the problem, students are required to explain their thinking and reasoning to their peers. In their monograph on the bar model method, Hong, Lim, and Mei (2009) stated,

Response A	Candice is correct because depending on the function if you have to take a square root there are two possible correct answers.
Response B	candice is correct because the two answers are part of the graph therefore they are correct
Response C	Jermaine, because the equations are linear, meaning only one intersect point

Fig. 7 Only one of these students fully understands the question.

Question 3 also was designed to get at the meaning of the solution of a system of linear equations. From the responses, it appears that only student C (see **fig. 7**) seems to understand the main point of the question—that two lines can intersect only in one point.

A Caution about These Templates

The templates presented here can be useful in giving teachers a place to start when writing open-ended questions, but teachers must be cautious when using them. Just because a question fits a template does not necessarily mean that the question is open ended or of high quality.

For example, we could ask the earlier question in this way:

> Jasmine solved $x + 3 = 5$ and got $x = 2$. Stuart solved $x + 3 = 5$ and got $x = 8$. Who is correct and why?

This form of the question is no different from asking the traditional question “Solve $x + 3 = 5$ for x.” The formulation does not involve the conceptual underpinnings of equation solving.

PREPARATION FOR LIFE

Teachers are under more pressure than ever to ensure that students perform well on standardized tests. Consequently, many are using more multiple-choice questions to prepare their students. School districts are using benchmark testing to assess students’ progress toward meeting standards and prepare them for accountability tests. These are all perfectly reasonable strategies, but mathematics education stakeholders must keep in mind the limits of these accountability tests. If we think about the purpose of schooling from a broader perspective and about preparing students to solve the kinds of problems that they will encounter in society—not just about preparing them for standardized tests—we need different strategies.

Open-ended questions can help teachers focus their instruction and assessment on NCTM’s Process Standards and on reasoning and sense making, which really is the heart of mathematics. Moreover, responses to open-ended questions give teachers so much more information about students’ ways of thinking and misconceptions, and these can provide important avenues for further investigation of mathematics. When students answer higher-order questions driven by the Process Standards and focused on meaning, they will be prepared for any test we give them—in school or in life.

ACKNOWLEDGMENTS

I wish to thank Tom Cooney, who taught me most of what I know about open-ended assessment. Much of the work presented in this article built on work published previously at www.heinemann.com/math (Cooney et al. 2002). I also wish to thank Ms. Yoder. Thanks as well to LouAnn Lovin for her feedback on drafts of this article.

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Response A	Create a system of linear equations that has the solution (−2, 3). Explain how you determined your system. $y=(x+2)^2+3$ ↑ horizontal shift, vertical shift
Response B	$y=-2x-1 \rightarrow y=-2(-2)-1$, $y=4-1 \rightarrow y=3 \rightarrow (-2,3)$
Response C	$y=x+5$ $y=-2x-1$ $x+5=-2x-1$ $3x=-6$ $x=-2$ I developed a system of linear equations with the solution (−2,3) then I set them equal to each other and solved for y.

Fig. 6 Students find it difficult to create a linear system when given the solution.

changing the direction of an inequality sign, an "equal to," and a "colored dot."

Template 2: Create an Example or a Situation

This form of question is similar to the form of the questions for the game show *Jeopardy*™. We give students some parameters and ask them to come up with an example or situation that fits the parameters. We give them the answer and have them come up with the question.

Some possible questions using this template follow:

1. Give a possible equation for the graph shown in **figure 5**. Explain how you determined your answer.
2. On a coordinate grid, plot and give the coordinates of four points that are the vertices of a rhombus. Explain how you know that your figure is a rhombus.
3. Create a list of ten different numbers whose median is 9. Explain how you know that the median is 9.
4. Give two complex numbers whose sum is $7 + 9i$. Explain how you know that your two numbers have the given sum.
5. Create a system of linear equations that has the solution (−2, 3). Explain how you determined your system.

The first time I used open-ended questions in my teaching, I included question 5 on an exam. Many students got every question correct except this one. The first section of the exam asked students to "solve these systems of linear equations by graphing"; the second section, to solve by substitution; the third section, to solve by elimination; and the fourth section, to solve by any method. Then I added this single open-ended question, and my students were thrown. I knew then that not only was I asking the wrong questions; I was also focusing my instruction on the wrong things. My students could follow procedures that I taught them, but they did not really know what a system of linear equations was or what a solution of a system of linear equations was.

Ms. Yoder's students' responses are informative (see **fig. 6**). Student A describes shifts of graphs of quadratic functions, whereas student B found a single line that contained the point (−2, 3). I think that students A and B would do just fine on a standardized test about systems of linear equations. Like my students who got every problem correct on my test except this one, these students might be able to answer standard questions without really understanding what a system of linear equations is. After reading these responses, however, I am much more confident that student C has a deeper understanding of systems of linear equations than either of the other two students.

Template 3: Who Is Correct and Why?

This form of open-ended question—Who is correct and why?—can be used to set up two opposing arguments. Then students can defend one or the other argument.

Some possible questions using this template follow:

1. Lucinda thinks that the grades in mathematics class should be calculated using the mean. Norm thinks that the grades should be calculated using the median. With whom do you agree and why?
2. Daniella is thinking about a particular quadratic function. Terry says that if Daniella told him the zeros of the function, he could tell her the equation of the function. Daniella maintains that Terry would need more information. Who is correct and why?
3. Candace said that if she solves the same system of linear equations as Jermaine, they could get two different answers and both be correct. Jermaine disagreed, saying that if they got two different answers, one of them must be incorrect. Who is correct and why?

"The model enhances their thinking and problem solving skills…Students are able to use the model to help them articulate and communicate their thinking and solution, and thus engage themselves in active collaborative learning" (p. 68).

Other combination approaches use elements of explicit instruction by teaching problem solving steps and strategies such as drawing a diagram, working backwards, and finding a pattern, in addition to constructing a bar to represent the problem structure. They use elements of a constructivist approach by allowing students to discuss, think, and reason as how to use these strategies to solve problems. After explicit strategies are taught, the teacher takes on the role of facilitator by guiding student's thinking using probing questions.

Typical math problem solving in elementary schools requires students to solve word problems. Many students find these problems difficult because the process used to solve them is complex. Teachers need to find effective instructional practices to help students solve mathematical problems. The purpose of this research project was to investigate the effectiveness of using a combination approach in which bar models were taught within the seven-step framework designed by Forsten (2010) compared to a combination approach teaching problem solving steps and strategies.

METHOD

To address the purpose of this study, a quasi-experimental design was used to compare the mathematical problem solving abilities of two groups of students. One group was taught to apply their mathematical skills by using the bar model method and the other group by using problem solving steps and strategies. A questionnaire was used to collect data on students' confidence levels using the problem solving method they were taught.

Participants

The participants in this study were 101 sixth grade students attending a suburban elementary school in northern Utah. These students included 57 males and 44 females and 90% were Caucasian with the remaining 10% other ethnicities. Twenty-two students were identified as economically disadvantaged. Students were from a predominantly white middle class suburban community. The teacher/researcher participated in the 2010 Singapore Math Bar Model training in Las Vegas, Nevada and used teaching methods as presented in the training.

Instrumentation

The teacher/researcher created problem solving pretest and posttest was administered to measure student problem solving progress. These tests consisted of word problems that aligned with the sixth grade core curriculum. Test problems required students to apply their knowledge of ratio and rate reasoning to solve real-world mathematical problems. These problems required students to use both single step and multiple steps to reach a solution. Students were asked to show bar models or problem solving strategies that were used to obtain their answers. Students were required to show all calculations used in solving each problem. Students were given the assessment prior to and at the end of a 3 week problem solving unit. At the end of the 3 week unit, students were also given a questionnaire which explored students' confidence levels in mathematical problem solving using the approach they were taught. The items on the questionnaire were adapted from a measure developed by Cummings, Lockwood, and Marx (2003). The questionnaire consisted of 6 questions, each of which had a four- point Likert response scale.

In solving this problem, students might use the midpoint formula to determine the coordinates of point B and then show that $AB^2 + BC^2 = AC^2$. In this way, they verify that triangle ABC is a right triangle because its sides satisfy the Pythagorean theorem and that, therefore, angle ABC is a right angle. Or, using the distance formula, students might show that $AE = AC$; then, using the side-side-side postulate, they can show that $\triangle ABC \cong \triangle ABE$. Therefore, $\angle ABE \cong \angle ABC$ because corresponding parts of congruent triangles are congruent. Because these two angles are congruent and form a linear pair, they must be right angles.

Still another way to solve this problem is to compute the slopes of $\overleftrightarrow{EC}$ and $\overleftrightarrow{AD}$ and show that their product is –1. More advanced students can demonstrate the dot product of [7, 5] (the rectangular vector from B to C) and [–5, 7] (the rectangular vector from B to A) is 0, making the two vectors orthogonal (perpendicular).

When students are required to provide multiple solutions, they often use a variety of representations. As they explain their reasoning, they are communicating. Although students need to rely on some procedural knowledge to answer this problem, they have to decide which procedures would apply to it. They are not provided with a step-by-step procedure; consequently, they are involved in problem solving as well as reasoning and proof. They are making connections among a variety of mathematical topics—slope, congruent triangles, midpoints, the distance formula, the Pythagorean theorem, and vectors.

WRITING OPEN-ENDED QUESTIONS

Open-ended questions can be written using various templates, several of which are discussed here. Teachers who are just beginning to use open-ended assessment can use these templates for creating their own questions. We provide examples of several types, and for one question of each type, we provide sample student responses.

Template 1: What's Wrong with This?

The earlier question about expanding $(x + 3)^2$ is an example of this type of question used to identify errors and misconceptions. We can ask students to identify errors and explain why they are errors. This template is useful for getting students to think critically about common misconceptions.

Some possible questions using this template follow:

1. Provide two different explanations as to why you cannot simplify the expression $(x + 3)/3$.
2. Bert was trying to graph $y = (x - 3)^2$. He said that he could simply shift the graph of $y = x^2$ three units to the left. Convince Bert that his method is incorrect.

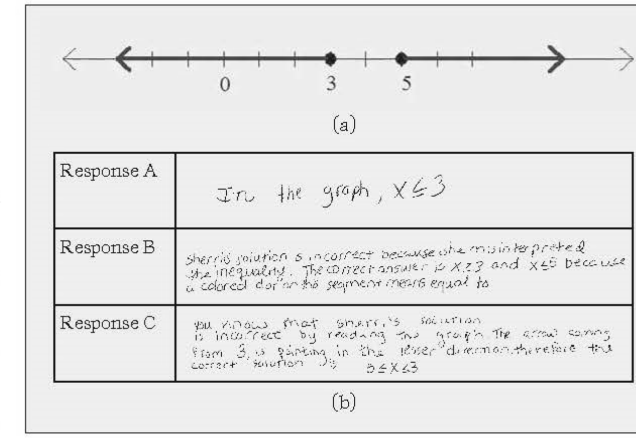

Fig. 4 Sherri's solution (**a**) is incorrect. Typical student responses are shown in (**b**).

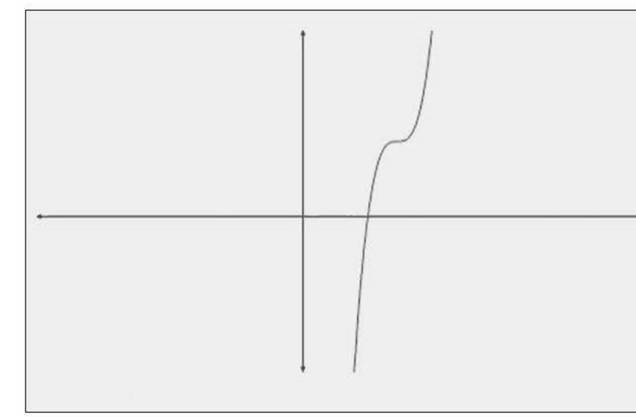

Fig. 5 Students are asked to provide a possible equation to match this graph.

3. Sherri claims that the solution set of the compound inequality $x \geq 3$ or $x \geq 5$ is shown in **figure 4**. Explain why Alaine's solution is incorrect. Provide the correct solution and explain how you know your solution is correct.

Question 3 was designed to counter the common student error of thinking that *or* always means that the arrows on the graph of a linear inequality should point in opposite directions. Of course, the correct solution set of the linear inequality is $x \geq 3$ because the *or* means one *or* the other *or* both. Therefore, any real number greater than or equal to 3 would be in the solution set.

None of the students who answered the question (even those whose solutions are not shown in **fig. 4b**) provided the correct solution. They focused on the direction of the inequality sign rather than on the meaning of the conjunction *or*. Student B appears to have some misconception about

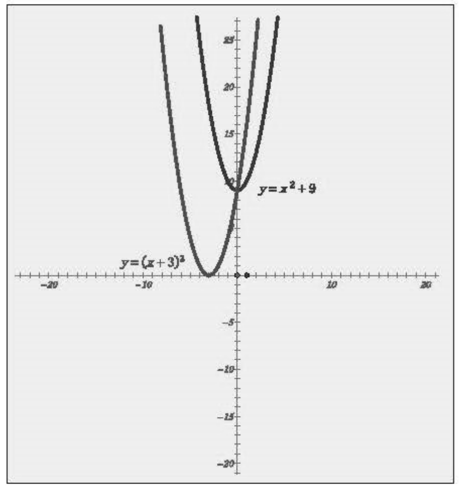

Fig. 1 Students might also argue that $y = (x + 3)^2$ and $y = x^2 + 9$ are, respectively, horizontal and vertical shifts of $y = x^2$.

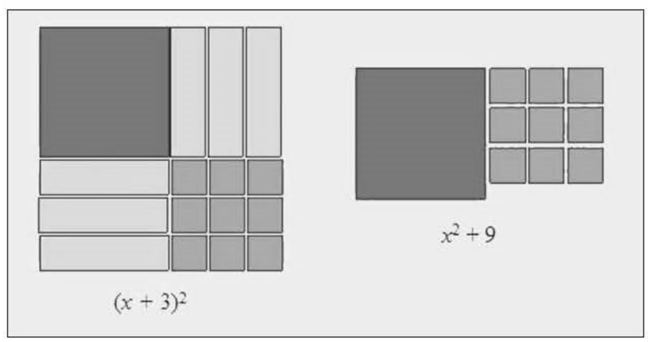

Fig. 2 Algebra tiles geometrically represent the statement $(x + 3)^2 \neq x^2 + 9$.

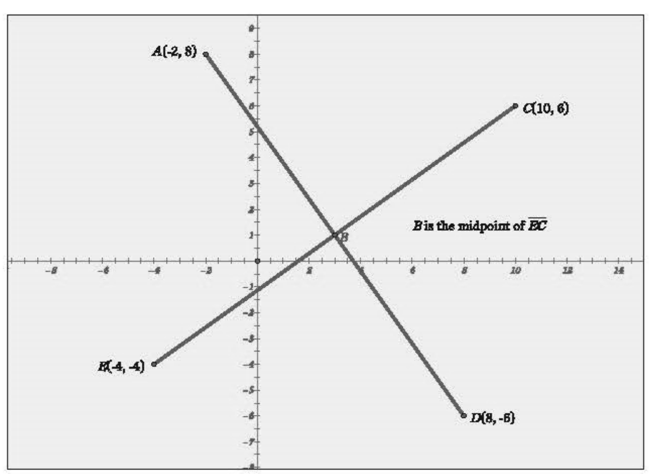

Fig. 3 Slopes, the Pythagorean theorem, congruent triangles, and dot products may all be used to show that $\angle ABC$ is a right angle.

OPEN-ENDED QUESTIONS CAN FOCUS INSTRUCTION ON PROCESS STANDARDS

Using NCTM's Process Standards as a guide, teachers can make questions more open and more focused on conceptual understanding.

Consider this traditional question:

> Expand $(x + 3)^2$.

We could revise this question in several ways. If we wanted to address the Communication Standard, we could ask students to explain how they determined their answer. We could take the question even further to incorporate other Process Standards. We could capitalize on a common student error and ask students to explain why $(x + 3)^2 \neq x^2 + 9$. Now we have expanded the question to include the Communication Standard and the Reasoning and Proof Standard. We could go even further to address the Representation Standard by asking students to give two or three different explanations of why $(x + 3)^2 \neq x^2 + 9$.

A typical first explanation that students provide is this:

> $(x + 3)^2$ means $(x + 3)(x + 3)$. I can use the distributive property to multiply these two binomials so I get $x^2 + 3x + 3x + 9$, which equals $x^2 + 6x + 9$, which is not the same as $x^2 + 9$.

Asking students for another explanation forces them to consider a different representation. For example, they might choose a numerical representation and substitute a numerical value for x. Their explanation might then be something like this:

> Let $x = 2$. $(x + 3)^2 = (2 + 3)^2 = 25$. $x^2 + 9 = 2^2 + 9 = 13$. Because $25 \neq 13$, $(x + 3)^2 \neq x^2 + 9$.

Students could also consider a graphical representation and show that the graphs of $y = (x + 3)^2$ and $y = x^2 + 9$ are different (see **fig. 1**). They could even consider the problem geometrically by using algebra tiles (see **fig. 2**).

If we teachers intentionally consider NCTM's Process Standards when writing questions, we can make sure that students are required to use the processes. With this particular question, we also counter a common student error in several ways. By seeing multiple representations, students are more likely to avoid the error later on.

What Process Standards might students use to solve the following problem?

> Use three different methods to show that $\angle ABC$ is a right angle. Explain your reasoning. (See **fig. 3** for solution.)

Procedure

There were four sixth grade classes at the elementary school. Classes rotated among the 4 sixth grade teachers for various subjects. In this study, students remained with their regular sixth grade class and participated in problem solving instruction one hour a day for three weeks.

All classes were taught to apply their knowledge of ratio reasoning to real world problems. During the first week, students were taught the concept of ratio. The students in the bar model group were taught to use Forsten's (2010) seven-step problem solving process. The students in the problem solving steps and strategies group used a modified version of Forsten's (2010) seven-step process but were taught a variety of strategies besides bar model drawing. The first week, students were taught to solve real-world problems that enforced the concept of ratio. The second week, students were taught to use ratio in finding rates. Unit rate problems and those that required students to find distance, speed, and time were solved. During the third week, ratios were used to solve problems requiring students to find percent. Single step and multiple step problems were taught.

Two classes were instructed using problem solving steps and strategies and the use of mathematical ratios. Two classes were taught using the bar model method, taken from the Singapore Math program. The students in the bar model group followed the seven step problem solving approach, developed by Char Forsten (2010). This approach had students: (1) read the entire problem, (2) rewrite the question in sentence form, leaving a blank for the answer, (3) determine who/what is involved in the problem, (4) draw the unit bars (5) chunk the problem, adjust the unit bars, (6) correctly compute and solve the problem, (7) write the answer in the sentence, and make sure the answer makes sense. The students in the problem solving steps and strategies group used a modified version of Forsten's (2010) seven-step process. In step 4, students chose a problem solving strategy instead of drawing a unit bar. In step 5, students chunked the problem and used the strategy to help them solve the problem. All practice problems required students' to show their bar models, ratios, strategies, and computation used to solve the problem. The same problems were taught and practiced in all classes. Students were given opportunities to collaborate with their peers while solving practice problems.

RESULTS

All sixth grade students were given a pretest prior to the 3 week instruction and a posttest after the 3 week instruction. The pretest mean score for students participating in the bar model group (N=53) was 24.45%. The pretest mean score for students participating in the problem solving steps and strategies group (N=48) was more than 11 points higher at 35.86%. The posttest for the bar model group was 86.17% which was higher than the posttest mean score for the problem solving steps and strategies group at 82.31% An initial analysis of the pretest means indicated that there was a statistically significant difference between the groups prior to instruction, therefore analysis of co-variance (ANCOVA) was used to control for differing pretest scores. The ANCOVA demonstrated there was a statistically significant difference between the bar model group and the problem solving heuristic group on the posttest when controlling for the

When we think about assessment in this era of No Child Left Behind, we often think about high-stakes standardized tests, which are typically multiple-choice tests. So much of what happens in mathematics classes is focused on preparing students to succeed on these tests. As I work with teachers, they express high levels of anxiety about making sure that their students are prepared for these high-stakes tests. Mathematics education stakeholders—including teacher educators, administrators, teachers, students, and parents—need to reflect on what standardized tests can and cannot measure. Even more important, they must evaluate the educational significance of those ideas that standardized tests cannot assess.

NCTM's Process Standards—Problem Solving, Reasoning and Proof, Communication, Connections, and Representation—are difficult to assess with multiple-choice tests. For example, one aspect of the Communication Standard requires students to "communicate their mathematical thinking coherently and clearly to peers, teachers, and others" (NCTM 2000, p. 60). This standard cannot be assessed through multiple-choice questions.

If we do not teach what is not tested, what are the implications of not preparing students to meet these Process Standards? Consider the following statement by a BC Calculus student:

> My experience in the past—and not to hate on the teachers I've had—but they've never really encouraged us to think. It's all been cookie-cutter questions, even with word problems. I remember my algebra 1 teacher—she had a little trick for everything. Of course, I don't remember the trick now, and I don't remember why I was doing it. I felt like there were a lot of shortcuts, and I was never really taught why we were using them. So I memorized everything, which is what I've been doing ever since (Stockton 2010).

This student was lamenting her inability to solve a complex problem. A student capable of handling the difficult BC Calculus curriculum expressed her own disappointment that the focus of her education had been procedural.

As teachers struggle to ensure that students are able to answer questions correctly on procedural tests, many are desperate to find ways to help them remember strategies and steps to find correct solutions. However, problems that people encounter in everyday life and careers rarely require rote application of procedures.

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OPEN-ENDED QUESTIONS AND THE PROCESS STANDARDS

Educating students—for life, not for tests—implies incorporating open-ended questions in your teaching to develop higher-order thinking.

Wendy B. Sanchez

All societies need citizens who can solve complex problems and apply knowledge in a variety of contexts as well as citizens who can work collaboratively to solve problems and communicate solutions to mathematics education stakeholders. We must educate students to use NCTM's Process Standards (NCTM 2000) and move beyond being able to work routine exercises on standardized tests. We are not educating students for tests; we are educating them for life. All stakeholders need to see this broader picture and support teachers in this broader purpose.

As a high school mathematics teacher and mathematics teacher educator, I have used open-ended questions as part of my own teaching practice. Open-ended questions, as discussed here, are questions that can be solved or explained in a variety of ways, that focus on conceptual aspects of mathematics, and that have the potential to expose students' understanding and misconceptions. When working with teachers who are using open-ended questions with their students for the first time, I have found that they learn a considerable amount, as I did, about what their students both know and do not know—much more than what they knew before they started using open-ended questions. Teachers are almost always surprised, a little disappointed, but often excited about what they discover.

I will share some student responses from the class of a high school mathematics teacher with whom I have worked. Ms. Yoder has high expectations of her students. Her students work together to solve problems that require a high level of cognitive demand; the kind of thinking necessary to solve the problems forces students to build "connections to underlying concepts and meaning" (Stein et al. 2009, pp. 1–2). After having her students work some of the problems presented here, Ms. Yoder commented, "I was dismayed at the lack of depth and the simplicity of some students' responses. I have always felt that I teach on a conceptual level, and I do a lot of listening to students' conversations to assure myself that the level of understanding meets my hopes and expectations. . . . But I have rarely required my students to write about mathematics." After using these problems with her students, Ms. Yoder reflected, "Asking these questions made me rethink my means of assessing students."

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pretest (F (1,98)=8.170, p<.005).

An addition analysis of gain scores was undertaken. The gain scores were derived by subtracting the pretest score from the posttest score. An analysis of the gain scores using a *t* test demonstrated there was a statistically significant difference between the bar model group and the problem solving steps and strategies group (*t*(99)=4.021, p<.000). On average the bar model group gained 17% more than the heuristic group. Comparisons of group mean scores are shown in Figure 1.

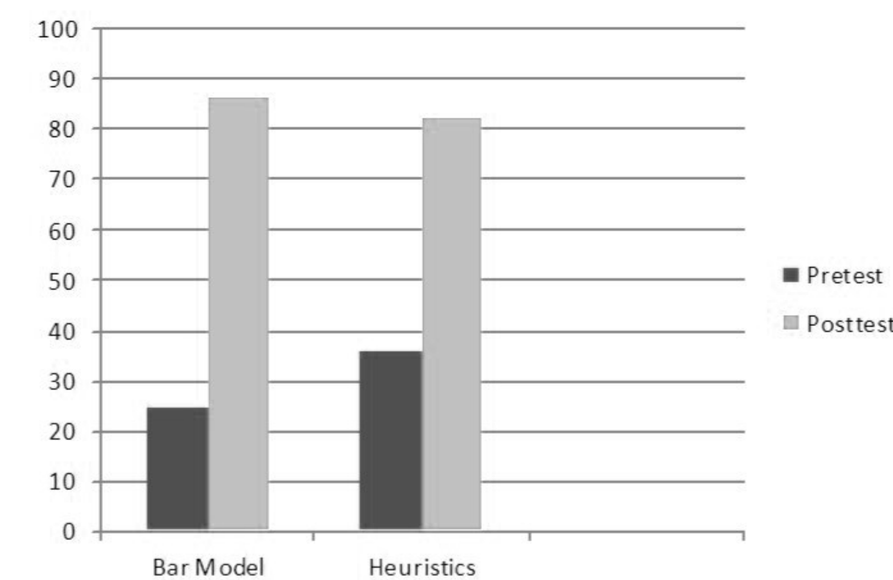

Figure 1. Comparison of group mean pretest and posttest scores

The problem solving confidence questionnaire mean scores for each group were analyzed for differences using a *t* test. The mean score for the bar model group was 19.49. The mean score for the Steps and strategies group was 20.23. A *t* test showed no statistically significance between the two groups. Students from both groups felt confident in their problem solving abilities.

DISCUSSION

The goal of successful mathematics instruction is to help students apply their mathematical skills in problem solving situations. This study sought to compare a combination approach in which bar models were explicitly taught with a combination approach in which problem solving steps and strategies were taught. This study sought to compare gains made by students taught using the bar model approach with those students taught using problem solving steps and strategies and compare confidence levels in problem solving abilities between the bar model group and steps and strategies group.

Data from this study expands research to show that when teaching ratio problems, the use of bar models were significantly more effective than using steps and strategies. Teachers should carefully consider what type of steps and strategies or model they use in their problem solving instruction. These should help students understand the structure of the problem. While it remains to be seen whether bar models would produce significant results in other problem solving areas, teachers should seek training in the use of bar models as another effective means to help students become successful problem solvers.

The majority of the students in both groups felt confident in their problem solving ability. The framework used and methods taught helped support students in their problem solving efforts and led to problem solving confidence. All students were taught to use the seven-step approach designed by Forsten (2010). This approach helped students navigate through the problem. It was the researcher/teacher's observation that most students had internalized this process by the end of the three week instructional period. This may have led 85% of the respondent to report they had a general approach

prime factors excluding repeated prime factors.

The two statements represent a reconceptualization of GCF and LCM based on the new definition of factor. **Figure 2** shows two strategies that students can use to find GCFs and LCMs using this reconceptualization.

Both strategies require first finding the prime factorizations of 150 and 84. Using a Venn diagram, the prime factorization of each number is written in its respective region, with all shared prime factors written in the overlapping region. The GCF is the prime factor combination in the overlapping region, and the LCM is the product of the prime factor combinations from all three regions of the diagram. Using a symbolic approach, the prime factorizations of each integer are written out symbolically. The GCF is the product of all of the shared prime factors (circled in **fig. 2**), and the LCM is the product of one of the integers and the unshared prime factors of the second integer. In both strategies, the GCF and LCM are thought of as products of combinations of prime factors.

A SURPRISING DISCOVERY ABOUT FACTORS

This rich activity allows students to explore complex mathematics and make a surprising yet useful discovery about factors. By viewing a number's factor as the product of *any* combination of the number's prime factors, students can work with number theory topics much more flexibly and efficiently. The activity described in this article can be modified for use with a wide variety of middle school students. For example, teachers can focus on numbers with relatively few factors and simpler prime factorizations for students with a limited understanding of prime factorization.

For students with more substantive prior knowledge, the teacher might consider including numbers with more factors or asking them to generalize using algebraic symbols (e.g., represent all numbers with 4 factors as the product of two primes, $p \times q$). Regardless of the modifications made, this activity can engage middle school students in important mathematical practices—exploring, explaining, justifying, and generalizing—around a central mathematical concept.

"Do the data you collected support your claim?" or "Can you find a number that goes against your claim?" remind students to think more carefully about their data.

CCSSM Practices in Action

SMP 3: Construct viable arguments and critique the reasoning of others.

SMP 7: Look for and make use of structure.

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Any thoughts on this article? Send an email to **mtms@nctm.org**.—*Ed*.

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Table 4 This listing gives all the factors of 24 using prime factor combinations.

Combination of Prime Factors	Factors of 24 (Products Are Listed Parenthetically)
0	1
1	2; 3
2	2 × 2 (4); 2 × 3 (6)
3	2 × 2 × 3 (12); 2 × 2 × 2 (8)
4	2 × 2 × 2 × 3 (24)

prime factorization. The generalizable nature of this argument is challenging for students to understand, so teachers should expect this discussion to last a significant amount of time (up to 30 or 40 minutes).

Once students are able to make more general arguments, the teacher should provide a variety of more complex examples for students to recognize that a factor of a number is a product of *any* combination of the number's prime factors. Ask the following questions:

1. Find all the factors of 24 using only its prime factorization.
2. Given $n = 2^5 \times 3^2 \times 7 \times 11$, decide if each of the following numbers is a factor of n: 2, 5, 6, 15, and 63.

Such questions force students to construct factors by multiplying various prime factors together. In past enactments, students would write the prime factorization of 24 as $2^3 \times 3$ and list different combinations of prime factors to find them all (see **table 4**). They would also recognize that if a number can be found in the prime factorization of n, then it must be a factor (e.g., 63 must be a factor of n because $3^2 \times 7$ is in its prime factorization). Justifying this claim has also led to interesting conversations about the associative and commutative properties. Providing additional composite numbers, such as 12 or 30, and asking students to find all their factors using only their prime factorizations was essential in helping them recognize how factors and prime factors are related. The end result was that students were able to identify all of a number's factors without even needing to know the number's value or doing any long division.

THE IMPACT ON GCF AND LCM

This activity can not only help students develop a deeper understanding of factors and their relationship to prime factorization but also support their understanding of other number theory topics. To illustrate this potential, consider the following problem:

> Find the GCF and LCM of 150 and 84.

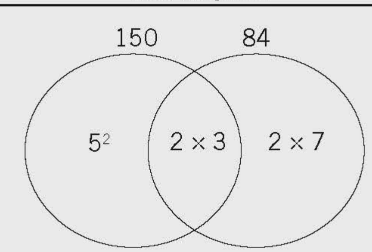
A typical solution strategy to find the GCF is to repeatedly divide each number by consecutive positive integers starting with 2 until the greatest counting number that divides evenly into both is found. To find the LCM, students usually make a list of consecutive multiples of each number and stop once they find a multiple that appears on both lists. Both methods are extremely tedious, prone to errors, and require little understanding of the meaning of greatest common factor or least common multiple.

By thinking of a factor as a product of any combination of prime factors, students can recognize that for any two integers—

a. the GCF is the product of all their shared prime factors, and
b. the LCM is the product of all their

Fig. 2 To find the GCF and LCM of 150 and 84, both a Venn diagram representation and a symbolic approach can be used.

Venn Diagrams



GCF = (2 × 3) = 6
LCM = $(5^2) \times (2 \times 3) \times (2 \times 7) = 2100$

Symbolic Approach

150 = (2 × 3) × 5 × 5
84 = 2 × (2 × 3) × 7
GCF = 2 × 3 = 6

150 = (2 × 3 × 5 × 5)
84 = 2 × (2) × 3 × (7)
LCM = (2 × 3 × 5 × 5) × (2 × 7) = 2100

they used when solving math problems.

Many students experience negative feelings during the problem solving process. Teachers can build problem solving confidence by teaching their students ways to work through and resolve these feelings. One way to do this is by giving students opportunities to collaborate with their peers. Students should understand that problem solving requires patience and may take time. Teachers need to allow adequate time for students to work through problems. Teachers can help students be successful problem solvers by supporting students with their problem solving instruction and providing a framework that helps students transition through the problem solving process.

If one point could be taken from this study, the researcher/teacher would hope it would be a better understanding of how to support and increase our students' ability to apply their mathematical knowledge in problem solving situations. International testing (NCES, 2007) shows that teachers in the United States need to find more effective instructional strategies that will help their students apply their mathematical knowledge in problem solving situations. Ultimately, it is not about numbers and comparisons but how well our students can think, reason, and apply their mathematical knowledge. This ability will affect their future educational opportunities and their place in a globally competitive world.

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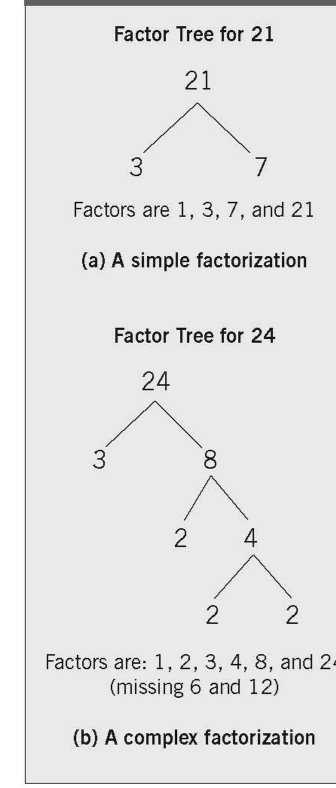
Table 3 Students record the prime factorizations of each number in the prime factorization table.

2 Factors		3 Factors	
Number	Prime Factorization	Number	Prime Factorization
2	2	4	2 × 2
3	3	9	3 × 3
5	5	25	5 × 5
7	7	49	7 × 7
11	11		
4 Factors		**5 Factors**	
Number	Prime Factorization	Number	Prime Factorization
6	2 × 3	16	2 × 2 × 2 × 2
15	3 × 5	81	3 × 3 × 3 × 3
21	3 × 7	625	5 × 5 × 5 × 5
8	2 × 2 × 2		
27	3 × 3 × 3		

Fig. 1 Although using a factor tree can be effective when the prime factorization is simple, it can obscure the composite factors of some numbers.



for the types of arguments students make. Comments such as "I know every number with 3 factors must be a perfect square because 9 is 3^2 and it has 3 factors, 1, 3, and 9" are quite common as students struggle to generalize their thinking beyond individual examples. Asking questions can help students make more general arguments about the relationship between the prime factorization of a number and its factors. Such questions might include the following:

1. "What do all the prime factorizations of numbers with three factors have in common?"
2. "Construct a new prime factorization of a number with 4 factors; what might it look like?"

For students who continue to struggle, the teacher should ask them to find the factors of a particular number using its prime factorization. The following represents a typical way that students have responded in the past:

> The prime factorization of 21 has 3 and 7, so both 3 and 7 are factors. Since a number is always a factor of itself, 21 is a factor. And, since 1 is always a factor of any number, 1 must be a factor of 21 as well. That's 4 factors!

Having students repeat this process for several numbers across one number type has helped many students begin to make a connection between prime factorization and factors. Such discussions meet the third Standard for Mathematical Practice: "Construct viable arguments and critique the reasoning of others" (CCSSI 2010, p. 6).

Exploring factor trees may also be a useful strategy to help students make this connection, but they should only be used for numbers with simple prime factorizations, such as 21 = 3 × 7, in which all the factors of 21 are visible in its factor tree (see **fig. 1a**). For more complex prime factorizations (e.g., $24 = 2^3 \times 3$), a factor tree should not be used because it can obscure some of the composite factors (see **fig. 1b**).

Once groups have begun to recognize the usefulness of prime factorization in identifying factors, the teacher should bring everyone back to a whole-class discussion to present their justifications. Typically, students use specific examples from their data, such as the explanation for 21 and its factors. The teacher can then ask the class to identify similarities across these specific examples. In past enactments, students used these examples to generalize their thinking to *types* of prime factorizations. For example, they would say that numbers whose prime factorizations can be represented as the product of two different prime numbers have 4 factors: the two primes, 1, and the number itself. This type of explanation promotes the idea that the factors of a number can all be found from the number's verify, and refute any conjectures. Teachers can also quickly assess how well students are able to find a number's factors and pause to review more productive strategies if students are struggling or using inefficient methods. If the class has already done a significant amount of work with factors and prime factorization, this part may be assigned as prehomework to the main activity.

Once groups find a few examples of each category, they begin to make conjectures about the types of numbers that have 2, 3, 4, or 5 distinct factors. For example, students often notice that numbers with exactly 2 distinct factors are prime numbers or that numbers with exactly 3 distinct factors are square numbers. (Note: This latter conjecture is incomplete.) Other claims are more difficult to make, such as the types of numbers that have 4 or 5 distinct factors. At this point, the teacher's role is to press students to support their claims *using the data*. As students provide empirical arguments, they can begin to make sense of their data. For example, if a student makes an incorrect claim (e.g., all square numbers have 3 distinct factors), others can check their data and provide counterexamples (e.g., 16 is a square number, but it has 5 distinct factors). As the teacher asks, "Do the data you collected support your claim?" or "Can you find a number that goes against your claim?" students will think more carefully about their data.

Once groups have had a few minutes to discuss their ideas, the teacher should facilitate a whole-class discussion in which each group presents its conjectures, resisting the temptation to reveal which conjectures are correct or incorrect. Instead, students should have the opportunity to comment on one another's ideas. Through debate, students can reflect on and revise their own conjectures (e.g., squares of prime numbers have exactly 3 distinct factors) and fill in any missing parts of their factor table. The entire class may or may not agree on a set of conjectures (see **table 2**); regardless, the teacher should instruct students to move on to the next part of the activity without revealing the correct conjectures.

Table 1 Students fill out the factors table with examples of numbers that have 2, 3, 4, or 5 distinct factors.

2 Factors		3 Factors	
Number	Factors	Number	Factors
2	1, 2	4	1, 2, 4
3	1, 3	9	1, 3, 9
5	1, 5	25	1, 5, 25
7	1, 7	49	1, 7, 49
11	1, 11		
4 Factors		**5 Factors**	
Number	Factors	Number	Factors
6	1, 2, 3, 6	16	1, 2, 4, 8, 16
15	1, 3, 5, 15	81	1, 3, 9, 27, 81
21	1, 3, 7, 21	625	1, 5, 25, 125, 625
8	1, 2, 4, 8		
27	1, 3, 9, 27		

Table 2 Correct conjectures are listed for the types of numbers that have 2, 3, 4, or 5 distinct factors.

Type of Number with No. of Distinct Factors	Correct Conjecture
2	All prime numbers have 2 distinct factors.
3	Squares of primes have 3 distinct factors.
4	Cubes of primes and products of two distinct primes have 4 distinct factors.
5	Squares of squares of primes (i.e., fourth powers of primes) have 5 distinct factors.

Part 2: Justifying Conjectures

In the second part of the activity, students justify their conjectures from part 1. While working in their groups, they first find the prime factorizations of the numbers in their factor table (see **table 1**)—typically by constructing factor trees—and record this information in a new table called the prime factorization table (see **table 3**). If necessary, the teacher can review the factor tree method beforehand to help students find the prime factorizations.

The teacher should instruct students to work with their groups to use the prime factorizations to confirm or refute their previous conjectures. The purpose of this part of the activity is to help students develop more robust and generalizable arguments that focus on the relationship between prime factorization and factors. When circulating from group to group, the teacher should listen

Using Games in Teaching Mathematics

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Games have been recognized as an engaging way of including student practice in mathematics instruction (Burns, 2009). With context, logic, variance, and possible competitive element tied to applications of mathematical concepts, games can quickly become effective and favorite mathematical format for children. Games are also a traditional activity across cultures, and *playing* has been recognized as one of six culture-driven universal mathematical behaviors (Bishop, 1991). Thus, games may serve not only as a vehicle for engaging in mathematical activities but may be connected to long-established cultural practices to recognize the mathematical contributions of cultural groups and their mathematical practices.

There is more than one way to define, describe, and implement an activity labeled as a "mathematical game" in the classroom. Traditional, time-proven games have pre-determined set of rules that have been established and followed for generations; they need to be familiar to all players and often may require long-term problem solving through sequencing a range of mathematical actions or procedures (see, for example, the probability games described by McCoy, Buckner, and Munley, 2007). Nowadays, a wide variety of web-based activities have gained popularity with children, parents, and teachers; some of these require completion of a single mathematical operation by a single individual in a rather repetitive manner. Still, the design involves a visually attractive, engaging colorful background, and there is often an inclusion of a character who may verbally or through movement guide the individual in the activity - and these are also considered and labeled as games. They are often available on electronic devices in classrooms and children gain access to them based on teacher-established criteria; sometime they allow for more than one player - see, for example, the games on Math-Play.Com (Popovici, 2014).

In this article, we take a specific approach to mathematical games. The focus is on game -like activities that could be developed by the teacher and are specific to the classroom and the students, the content being taught, and the specific lesson or unit objectives. These activities are geared to developing mastery of specific concept or skill in a discourse-rich environment that allow for a variety of practice scenarios of a newly acquired mathematical concept or skill; for example finding possible addends for certain sums, finding all possible factors or factorization expressions for composite numbers, or finding patterns in the multiplication table. We will also review a process for teachers to develop and implement such activities in order to respond to the immediate needs in the classroom while using them as an effective differentiation tool.

Quality Mathematical Games

A key component of quality mathematical games are the problem-solving features they naturally possess, together with their potential for posing worthwhile mathematical tasks. In a NCTM problem-solving research brief, Cai and Lester (2010) outline the following criteria to determine and guide the use of worthwhile mathematical tasks:

1. The problem has important, useful mathematics embedded in it.
2. The problem requires higher-level thinking and problem solving.
3. The problem contributes to the conceptual development of students.
4. The problem creates an opportunity for the teacher to assess what his/her students are

make sense of important properties such as the distributive, associative, and commutative properties. Factors also support students' ability to find least common denominators when adding and subtracting fractions. They also play a special role in helping students understand why a fraction's decimal representation terminates or repeats. In the secondary grades, students can use factors to simplify algebraic expressions. For example, factoring polynomial and rational expressions allows students to identify relationships between their factors and zeros; they can then use this information to identify intercepts, extreme values, end behaviors, and asymptotes when graphing them on a coordinate plane (CCSSI 2010).

Another important concept closely tied to factors and factoring is *prime factorization*. Typically, students learn to find the prime factorization of a number in the middle grades. This important concept is expressed by the fundamental theorem of arithmetic, which states that each positive integer greater than 1 can be factored into a unique product of prime factors (i.e., its prime factorization). For example, the number 72 can be decomposed into several different factorizations (e.g., $2 \times 3 \times 12$; 8×9), but assuming that order does not matter, it can be written in only one way using only prime factors (i.e., $2^3 \times 3^2$).

Although there is little research on middle school students' understanding of number theory topics, research with prospective elementary teachers suggests that working with prime factorization can support a deeper understanding of divisibility, a greater ability to solve and make sense of GCF and LCM problems, and a more meaningful conceptual understanding of algebra concepts (Brown, Thomas, and Tolias 2002; Zazkis and Campbell 1996). The little research that does exist on middle school students' understanding of number theory topics supports these findings (Zazkis and Gadowsky 2001).

Research with prospective elementary teachers suggests that working with prime factorization can support a more meaningful conceptual understanding of algebra concepts.

Despite its importance, students struggle to use prime factorization even when they know how to find it. For example, when they are asked if $m = 3^3 \times 5^2 \times 7$ is divisible by 7, many students compute the value of m first and then divide it by 7 using long division; they do not recognize that 7 must be a factor of m because it is part of m's prime factorization (Zazkis and Gadowsky 2001). Two explanations for these difficulties are that students (1) usually think of a factor as one of two integers multiplied together and (2) are not accustomed to working with prime-factored representations of natural numbers. By giving students more opportunities to explore numbers in prime-factored form, they can improve their ability to work with prime factorization (Zazkis and Gadowsky 2001).

DESCRIBING THE ACTIVITY

The primary goal of this activity is to help students change the way they think about factors by helping them understand that a factor of a number can be viewed as the product of *any* combination of the number's prime factors. Depending on students' prior knowledge of factors and prime factors, this activity can last several (2–5) class periods. It is suggested that lessons leading up to this activity teach students how to find factor pairs of a number, define prime and composite numbers, and construct factor trees to find the prime factorizations of positive integers. The activity has two main parts: *making conjectures* and *justifying conjectures*.

Part 1: Making Conjectures

To begin the activity, the teacher asks students to work in small groups of three or four to find several examples of numbers that have exactly 2, 3, 4, or 5 distinct factors. When circulating from group to group, the teacher should look to see which methods students use to find the factors of a number. Some might use trial division, whereas others may generate lists of factor pairs. As each group identifies different positive integers with the required number of factors, they record these numbers in the factors table (see **table 1**). Some students struggle to identify numbers with exactly five distinct factors because such numbers are quite large. If they cannot find any examples, the teacher should encourage them to move on because they will have opportunities to fill in the table later.

The first part of the activity can be beneficial for students because it allows them to review previously learned methods for finding factors while generating some data about numbers and their factors. As the activity continues, students will refer to their table to help them construct,

Rethinking Factors

Ziv Feldman

Help students develop an entirely new way of thinking about factors.

How do you think your students would define the term *factor*? Many learn that a factor of a number is an integer that can be multiplied by another integer to get the number (e.g., 4 is a factor of 20 because $4 \times 5 = 20$). Another view is that a factor of a number is an integer that divides evenly into the number (e.g., 4 is a factor of 20 since $20 \div 4 = 5$ with no remainder). Unfortunately, these definitions limit students' understanding of factors and ability to solve more challenging problems, such as this example: Is $2^5 \times 3^2 \times 7 \times 11$ divisible by 63?

This article describes an exciting exploration-based activity in which students develop an alternative definition of factor that can help them solve problems like the one presented above. Students work in groups to collect data, analyze the data to make conjectures, and then spend a significant amount of time debating and justifying their conjectures. Although designed for prospective teachers (Teppo 2002), the activity can be modified for middle school classrooms. Details are given about how to implement the activity in a middle school classroom as well as what we learned from past enactments with prospective elementary teachers. It concludes with a discussion of the activity's potential to impact students' related work with greatest common factors (GCF) and least common multiples (LCM).

WHY ARE FACTORS IMPORTANT?

Factors are part of a larger field of study called *number theory*, which includes topics such as prime numbers, composite numbers, divisibility, prime factorization, greatest common factors, and least common multiples. According to the Common Core State Standards for Mathematics (CCSSM), students should be learning number theory topics from kindergarten through high school (CCSSI 2010). In fourth grade, for example, students find factor pairs of whole numbers less than 100 and recognize that a whole number is a multiple of each of its factors. In sixth grade, they compute the GCF and LCM of two whole numbers. In high school, they factor quadratic and other polynomial expressions.

Factors and factoring concepts support a wide range of mathematical topics across grade levels. They help students identify the relationship between a factor and its multiples and

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learning and where they are experiencing difficulty.

5. The problem can be approached by students in multiple ways using different solution strategies.
6. The problem has various solutions or allows different decisions or positions to be taken and defended.
7. The problem encourages student engagement and discourse.
8. The problem connects to other important mathematical ideas.
9. The problem promotes the skillful use of mathematics.
10. The problem provides an opportunity to practice important skills. (p. 1-2).

A well designed and understanding-driven mathematical game-like activity should meet at least a subset of the criteria, and should naturally direct students through applications of the problem solving process in ways that they find engaging and motivating. Some of the key features to achieve these characteristics mean that the game task:

1. Prompts and allows students to apply different strategies to reach a successful move (equivalent to a correct solution of a mathematical task) in the game. The teacher should encourage the use of a variety of strategies rather than require – or imply – the use of a single strategy or procedure. Students could either be provided a set of tools – for example, manipulatives and/or paper and pencil to be encouraged to draw and write while solving the problem, or they could be encouraged by the teacher to seek solutions using different mental strategies. For example, consider the variety of ways for students to find the missing addend to make 10 when the student draws 3 as a first addend: from using a ten frame with or without manipulatives to mentally adding on from 3.

2. Stimulates student reasoning and demonstration of understanding, and prompts them to make connections with other mathematical knowledge and experience that may not be immediately obvious in the game task. One way to achieve this would be to require students to explain their own game move to their peer and refer to past experiences and knowledge that justify their action – for example, verbally explain that they started with three and counted seven more to make a ten as that is how they did it with the ten frame in class.
3. Requires students to understand and interpret their own and their peer's decisions and results. Asking students to agree with or confirm the validity of the game move performed by their peer would be appropriate here.

Well designed games result in a range of student benefits. In a game-like format, students may be more likely to tackle challenging mathematical tasks and exhibit their mathematical thinking. A mathematical task embedded in a game should encourage and naturally trigger students' willingness to verbalize and share justification of a game move or comments on their own or their peers' mathematical actions. The potential of learning from peers is intrinsic to a well designed game-like task; students could expand the range of mathematical strategies they know and apply by observing the moves of their peers and/or partners or by considering game moves that may lead them to success in the game. Future references to the game tasks could lead students to the discovery of other mathematical principles that will advance their mathematical competencies.

Mature proportional reasoning is indicated by the successful navigation of a variety of problems from diverse contexts.

Correct solutions will distinguish between students using build-up and multiplicative strategies.

- *4 matchsticks:6 paperclips = 14 matchsticks:x paperclips.* This problem has a scale factor of 3.5. Because this factor is more than 1 full repeat, you can determine how students deal with "leftovers." Which students have a basic understanding of the build-up strategy, and which use techniques that are more sophisticated?
- *4 matchsticks:6 paperclips = x matchsticks: 9 paperclips.* This problem is similar to the classic problem but removes the potentially distracting second appearance of the 6 and allows you to determine which multiplicative strategy your students are using. Because the scale factor is 1.5, you may see some students lured back into additive thinking.
- *6 matchsticks:12 paperclips = 22 matchsticks:x paperclips.* This problem uses a "messy" scale factor of 3 2/3. Which measure space ratio do your students use? The *between*-measures ratio clearly leads to a more efficient solution, but you may still see students scaling by the *within*-measures ratio. Use this opportunity to compare and discuss solution strategies and the multiplicative relationships in a proportion.

Once you know how your students are thinking about proportions, you can guide them to explore all aspects of the multiplicative structure of proportion. As they expand their repertoire of solution techniques, they strengthen the connections among division, fractions, and rational numbers, and they lay the groundwork for working with slopes and rates of change in functions. You can help your students navigate their routes to reason.

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Any thoughts on this article? Send an e-mail to ***mtms@nctm.org***.—*Ed*.

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Competition or collaboration.

Games could be competitive, or may provide a non-threatening, individually-paced environment for practice where students are only expected to apply correctly mathematical principles while communicating with other students. In competitive games, score keeping may be tied to the learning or practice of mathematics, by having students calculate their own scores, or by tying mathematical procedures and practice in the score determination. Regardless of the type of game selected, the focus should be on the mathematically important outcomes achieved through a variety of strategies shared with others and mastered through a sequence of attempts that solidify students' understanding and ability to apply mathematical ideas in practice.

A well designed and implemented game-like math activity should have the potential to be extended for more – and more sophisticated – practice and for extending the mathematical content to more advanced concepts and skills. Alternating competitive, collaborative, individual, or team games would provide for differentiation of the mathematical tasks and variety in the way students approach them. As games could be one of the ways to change attitudes, perceived abilities, and beliefs toward mathematics, competitive games should be used occasionally until students feel more confident in their mathematical skills. Games should not negatively influence by associating the game loss with lack of individual skills that sometime may be reinforced by a lack of success in a game. Occasionally, competitive games that require strategic moves (for example, having a choice to move the game piece in one direction that would influence the number of moves needed to finish first versus earning more points in selecting the other direction) could provide students with opportunities for linking their choice with their outcome while still practicing the important mathematics.

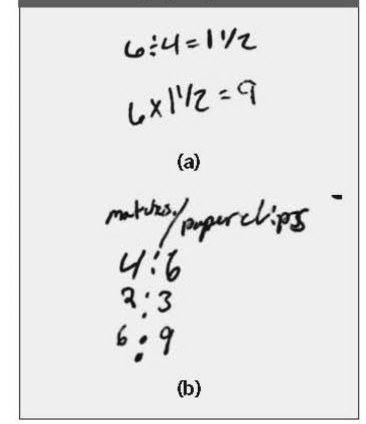
Role of Student

As with any mathematical task, it is imperative that students understand the objective and the expected outcome of the game-like activity they are to tackle. Repeating the instructions of the game to a peer is one way to check for such understanding; students could also perform a trial run and come back with any questions. If the teacher decides that a model will be best in providing students with solid understanding of the game procedure and rules, the modeling should be done by students and not by the teacher. To be instructionally effective, good math games require student involvement and investment. Students should be kept accountable for their game moves; this can be achieved by peer checks or by student recordings of their or their opponent's game moves. Thus the teacher can check scores and the accuracy of students' mathematical thinking and procedures applications, and know if and how much students practice within the game format. Some of these could be provided in the game rules – for example, students need to make 5 tries each. Similarly, a game becomes a good differentiation tool in terms of student practice – some students may need fewer game tries with smaller numbers while others will advance more and work with an additional challenge within the same game format.

Role of Teacher

The role of the teacher when mathematical games are played in the classroom is critical. Teachers should actively participate throughout the game time by observing, listening, differentiating, and providing input if necessary. The variety of formative assessment information

Fig. 7 Multiplicative strategies are used by these students, but it is unclear into which subcategory they fall.



grade to 29 percent in eighth grade (see **table 2**). When students annotated their work, the *within*-measures multiplicative strategy was preferred.

We believe that many students begin with a build-up strategy, in which the tendency is to compute how to transform the given number of matchsticks into the target number and then duplicate the operation for the paperclip quantity. The transition from repeated addition to a scaling operation (a *within*-measures strategy) is natural. Prior research, although not conclusive, supports this idea (Steinthorsdottir and Sriraman 2009). Problem context, number structure, and other factors also affect student choices, and we perhaps define the *between*-measure-space strategy more narrowly than do other researchers.

WHY IS THE SUCCESS RATE LOW?

Older students correctly solve this problem more frequently than younger students, but their success rate is surprisingly low. Our numbers are similar to those reported by Karplus, Karplus, and Wollman (1974). For example, 29 percent of our 412 students (grades 5–8) correctly solved the problem, and 44 percent used additive reasoning. Karplus, Karplus, and Wollman (1974) report that 37 percent of 610 students (grades 4–9) used correct reasoning and 32 percent used additive reasoning.

It is important to consider what makes this problem harder than expected. The Mr. Tall and Mr. Short problem uses a scaling context, which Lamon (1993) argues is difficult. Although the numbers involved are small, both the *within*- and *between*-measure-space ratios are 1 1/2. Research is clear in stating that integer relationships are easier than non-integer relationships, and yet one would expect that 1/2 relationships would be the next easiest. This does not appear to be true for the 1 1/2 relationship because, we argue, 1 1/2 is less than 1 full repeat.

Consider, for example, the problem 4:6 = 14:*x*, in which the scale factor is 3 1/2. Students with a beginning understanding of proportion can build up using whole units: 4:6, then 8:12, and then 12:18. The target is not quite reached. Some students will partition 4:6 to correctly finish the build-up (2:3 joined to 12:18 gives 14:21). Other students will fall back on additive thinking to finish the problem, adding 2 to both quantities and getting the incorrect answer of 14:20. Two levels of understanding are uncovered in this case.

With a scale factor of 1 1/2, students must immediately deal with the fractional part. In this case, students who can build up with integer repeats but fail when faced with "leftovers" are indistinguishable from students who are thinking additively.

The small numbers in this problem also mean that the relative difference (6 ÷ 4 = 1.5) and absolute difference (6 – 4 = 2) between the numbers are approximately the same. The incorrect additive answer is thus "in the ballpark" and may not alert students to an error. The fact that 6 appears as a measurement in both ratios may also be a source of confusion.

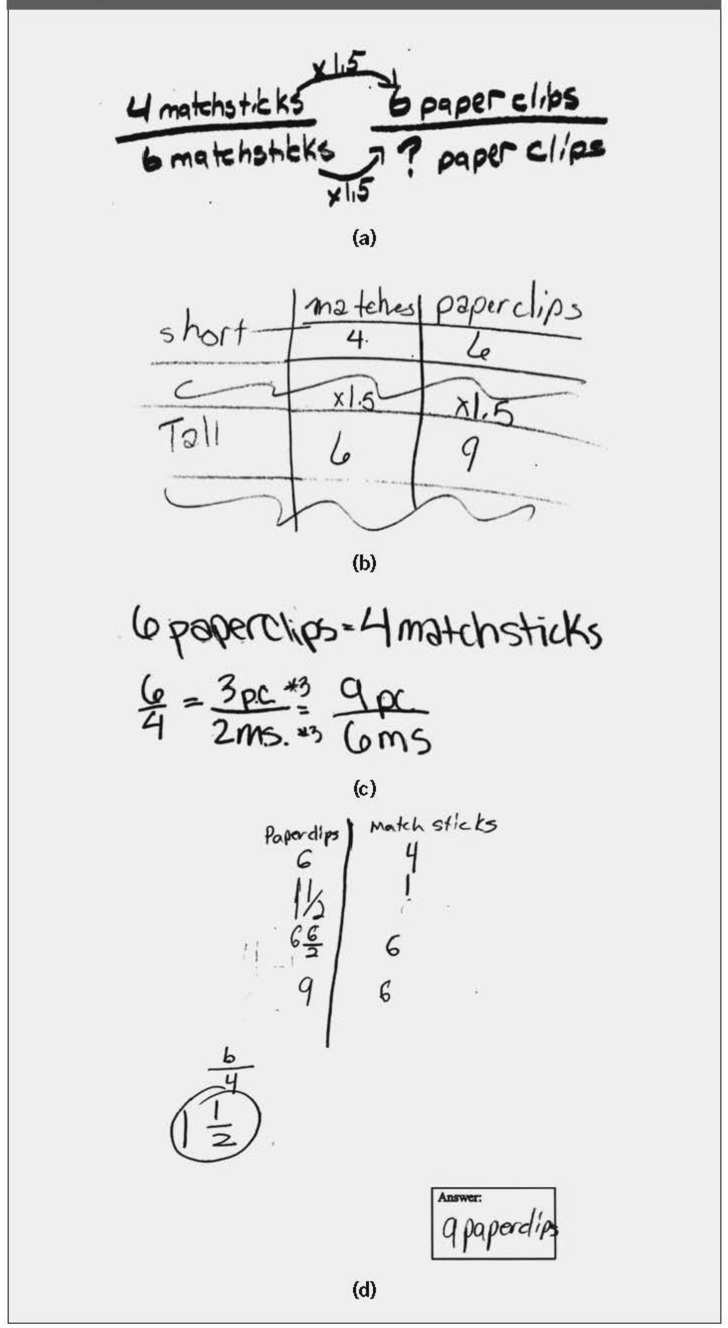
SHOULD YOU ACCEPT THE MR. TALL AND MR. SHORT CHALLENGE?

No one problem can fully assess proportional reasoning. In fact, mature proportional reasoning is indicated by the successful navigation of a variety of problems from diverse contexts, no matter the complexity of the number structures involved. On the other hand, teachers do learn much from a careful analysis of a single problem and can use the resulting information to make instructional decisions.

We recommend trying one of the following variations of Mr. Tall and Mr. Short to delve more deeply into your students' thinking. First, consider which broad category of reasoning (illogical, additive, build-up, or multiplicative) you expect most of your students to use. See **table 1** and note that **table 2** suggests that you may see all reasoning categories in your classroom. Then choose a variation that will enable you to verify and refine your assessment. The first two problems are appropriate for novice students and are designed to reveal a range of sophistication in build-up strategies. The last two problems should allow you to assess how robust and flexible your students are in the use of multiplicative strategies. The classic problem (see **fig. 1**) may be most illuminating in assessing experienced students. You may discover that some students fall back into additive thinking, and you will thus be able to address their misconceptions.

- *4 matchsticks:6 paperclips = 20 matchsticks:x paperclips.* This problem uses a whole-number scale factor. This simple case can launch discussions on relative versus absolute comparisons.

Fig. 6 These student-work samples demonstrate various multiplicative strategies, either using the *within* or *between* ratios.



student uses the factor of 1.5 with the unwritten units *paperclips/matchsticks*, and so is employing the *between*-measure-space ratio.

On the other hand, the *within*-measures strategy compares matchsticks with matchsticks and paperclips with paperclips. Students determine a scale factor, perhaps in one step (6 matchsticks ÷ 4 matchsticks; see **fig. 6b**) or after reducing the original ratio of matchsticks to paperclips to 2:3 or 1:1.5 (see **figs. 6c** and **6d**). The scale factor is then applied to Mr. Short's paperclip measurement to find Mr. Tall's measurement. Note that the extra step of reducing the original ratio results in a simpler integer scale factor, and thus, finding the unit rate does not necessarily imply the use of the *between*-measure-space relationship.

The number structure of this problem makes it difficult to determine which relationship is used unless the students label quantities. In **figure 7a**, the student uses a multiplicative strategy, but it is not clear whether the 6 represents matchsticks or paperclips. This type of work is coded as multiplicative-ambiguous in **table 2**.

For other responses, we cannot determine whether the students used a build-up strategy or a multiplicative strategy. The strategy in **figure 7b** is an example. It is clear that the student reduced the given ratio 4:6 to 2:3, but it is unclear whether the student then found the equivalent ratio of 6:9 by adding the ratios 4:6 and 2:3 or if the student applied the scale factor of 3 to the 2:3 ratio. This type of work is coded as ambiguous in **table 2**.

The percentage of students using a correct strategy increased from 13 percent in fifth grade to 16 percent in sixth grade, to 27 percent in seventh grade, and to 45 percent in eighth grade. As expected, the use of multiplicative strategies increased by grade level, from 2 percent in fifth

a teacher could collect during the game is invaluable for future instructional decisions. Teachers should demonstrate engagement and interest in the game and its outcomes to signal their own investment as part of the instructional process.

Teachers should employ good questioning techniques when necessary to provide feedback or clarifications for students; this will help students to continue utilizing problem solving steps even if conflicted or uncertain about a next step of the game. The teacher should encourage this process by posing questions that further students' learning and understanding rather than provide immediate answers. Often, students will correct their thinking within the questioning process; if necessary, the teacher could provide clarification, or pose a similar simpler task to trigger recollection and prompt new ideas.

Teachers should capitalize on the opportunities for students to engage in the core mathematical practices while involved in games. The Utah Core mathematics curriculum sets forth eight mathematical practices. Students in grades K-12 are expected to engage regularly in these practices and to employ them effectively in all of their mathematical learning and problem solving. These practices are:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.

8. Look for and express regularity in repeated reasoning. (UEN, 2011)

These mathematical practices are key to the learning of mathematics while children develop their knowledge, skills, and attitudes toward mathematics. Teachers can target one or more of the practices through games by promoting them through the design they utilize; however a game would be most effective if it requires application of a combination of most of the practices in a single game.

Game-like Activities Design

Every game used in the classroom for teaching, learning, and practicing mathematics should be aligned with the instructional objectives of the lesson and the unit. This alignment allows for teachers to design their own games based on the relationships and concepts they teach. Some guidelines help with decisions regarding content and context of the games:

1. Games do not have to be complicated; the less complex the rules are the less time students will need to understand what they are expected to do and the more time they will spend practicing.
2. If students have not experienced practicing with games before, try a game in a content they already have encountered. A review of the last year content may be a great starting point for the school year Start with a simple game you know; adapt it to work for your instructional objective, and start!
3. "Recycle" known and used games by changing operations and numbers. Teachers can determine the ranges of numbers students use, and they can use them to differentiate the task with different levels of difficulty.

recognize the need to compare quantities, although many students choose the incorrect type of comparison. In the next section, we conjecture that this specific problem has features that lure students into additive thinking when they might not use it to solve a different problem.

In using a build-up strategy, students recognize that the ratio of Mr. Short's 4 matchsticks to 6 paperclips forms a unit that needs to be coordinated. This unit (the *between*-measure-space ratio) can then be joined repeatedly to the same ratio, joined to an equivalent ratio, or partitioned (Lobato et al. 2010). In each case, the result will be an equivalent ratio. Students use this knowledge to generate equivalent ratios of 4:6 until a desired ratio is found.

Figures 5a and **5b** are examples that show how the total number of paperclips is determined by computing the number of extras needed by Mr. Tall to account for his 2 extra matchsticks. In **figure 5a**, the rate of 1.5 paperclips/matchstick is added to the original ratio 6:4 twice. In **figure 5b**, the unit rate is computed (using the figure), and then the equivalent ratio 2:3 is added to the original 4:6 to reach the ratio 6:9. In both cases, students join equivalent ratios to create a newly composed ratio of 6 matchsticks to 9 paperclips. In **figure 5c**, the student uses 6 iterations of 1.5:1, one for each matchstick, to create a new composed ratio of 6:9.

Multiplicative strategies explicitly use one of the proportion's multiplicative relationships (either the *between*-measure or *within*-measure space). The *between*-measures strategy (comparing paperclips with matchsticks) is applied when students explicitly use the fact that Mr. Short's measurement is 1 1/2 times greater in paperclips than in matchsticks (or 2/3 as much in matchsticks as in paperclips), and hence so is Mr. Tall's. In **figure 6a**, the

Fig. 5 The number of paperclips is determined by employing the build-up strategy.

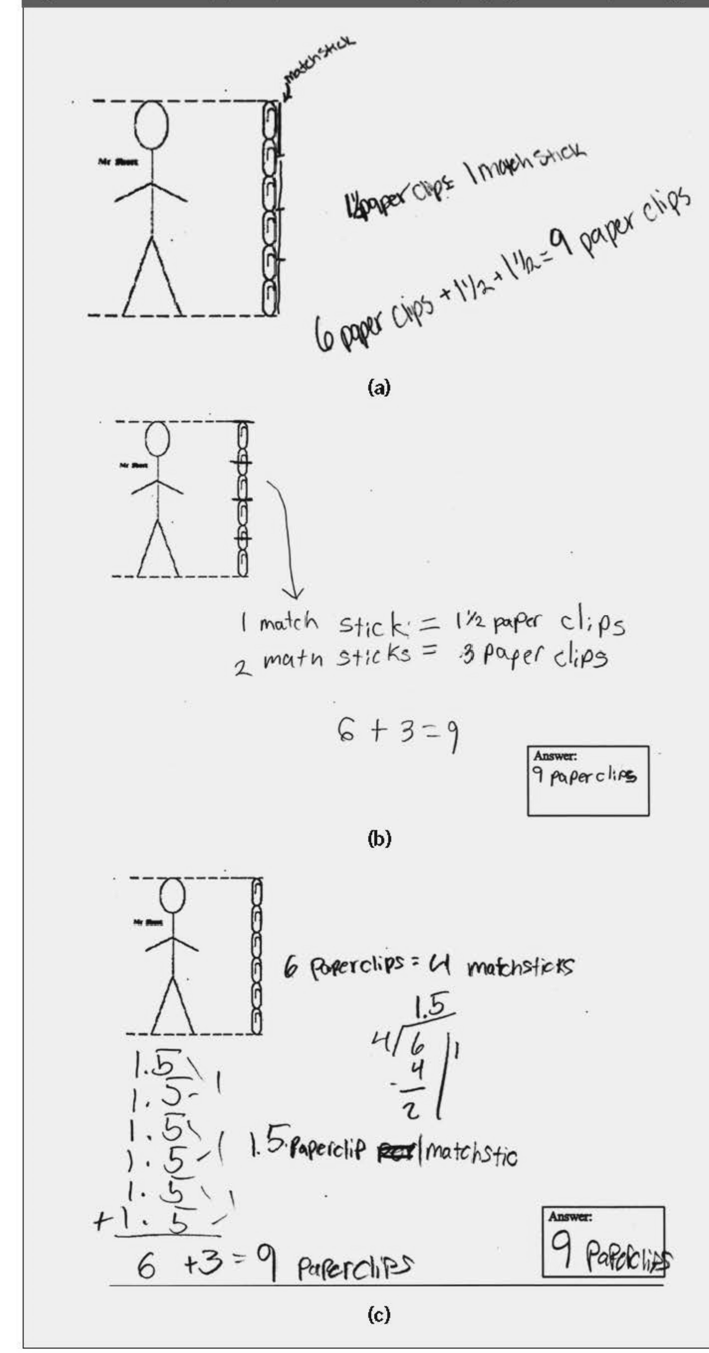


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Fig. 3 These samples of student work demonstrate illogical strategies.

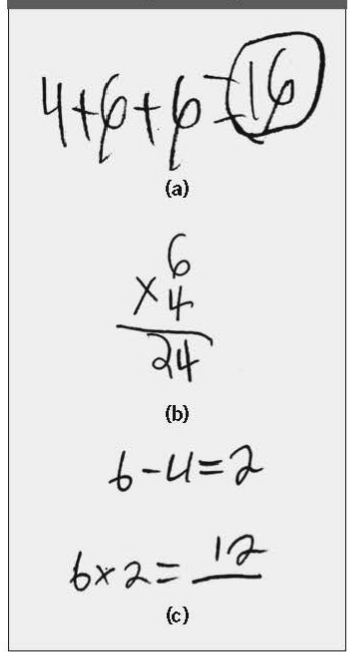
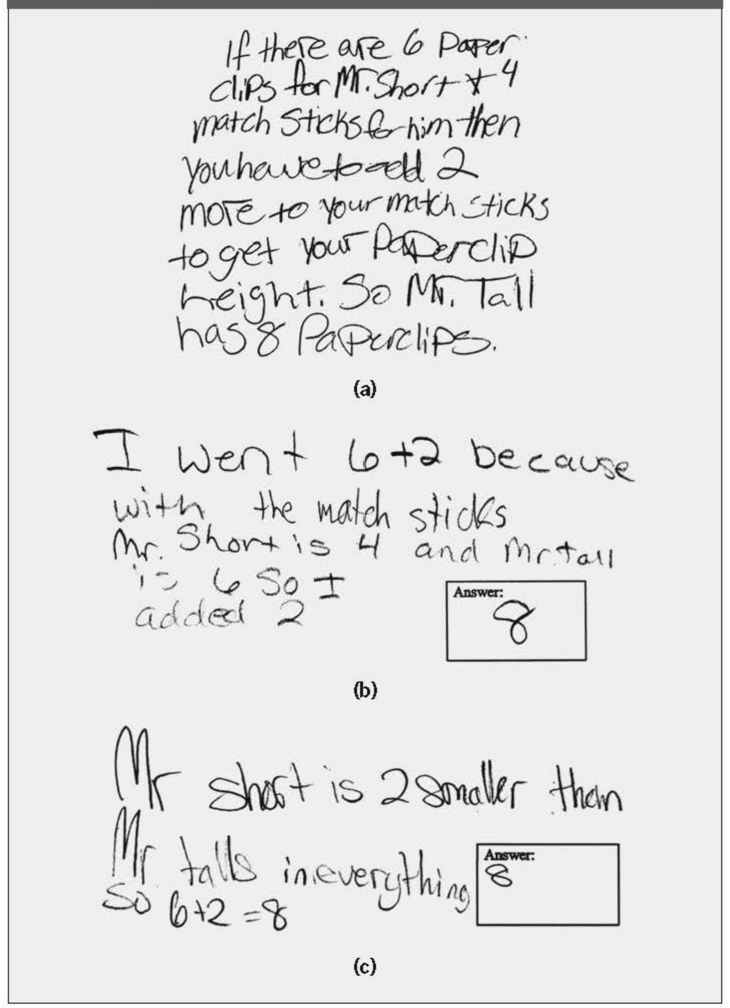


Fig. 4 With these additive strategies, students were using absolute comparisons rather than relative comparisons.



shows little understanding of proportions. Numbers given in the problem may be combined haphazardly, as in **figure 3a**, in which the three numbers are summed, or in **figure 3b**, in which two numbers are multiplied. Also classified as illogical is the work in **figure 3c**; in this example, the student seems to compare two quantities additively and then applies the result multiplicatively.

In the additive category, students use absolute comparisons rather than relative comparisons. **Figure 4a** shows how a student computes the *between*-measure-space difference (Mr. Short is 6 paperclips or 4 matchsticks tall, which are 2 apart) and maintains that difference for Mr. Tall. In **figure 4b**, the student instead computes the *within*-measure-space difference (6 matchsticks for Mr. Tall compared with 4 for Mr. Short) and answers that Mr. Tall's height in paperclips is 2 more than Mr. Short's. **Figure 4c** illustrates a strategy in which a student declares that there is a difference of two "in everything," and, therefore, Mr. Tall is 8 paperclips in height.

As **table 2** shows, although the majority of students reason incorrectly, the distribution of errors changes with grade level. Between the fifth grade and the sixth grade, answers falling into the illogical category decreased from 48 percent to 23 percent, whereas those falling into the additive category increased from 39 percent to 60 percent. For seventh grade and eighth grade, the percentage of illogical answers is essentially the same as that for the sixth grade, whereas the percentage of additive answers declines to 50 percent and 33 percent, respectively. This may indicate that more mature students

4. Vary larger group games with games in pairs, and even use the pairs as a practice for a more challenging group version of a game. The more variety in input students receive, the more versatile their own methods of solving mathematical problems may become.
5. As a teacher, be prepared to alter some of the rules or make other needed changes if you observe that the game is not understood by the students or is not aligning well with your instructional objective. Value the instructional time and make adjustments as needed. Even if you need to direct the students to a different task instead if finishing the game, do so; you may bring the game or a version of it on one of the following days when students will be better equipped with solid mathematical knowledge to be efficient and successful problem solvers.

How to Develop Instructional Game-like Activities

1. Start with your current Core domain and standard and the respective unit goals and objectives. They will be your indicator for the key mathematical skills and knowledge students will need to master. While work with numbers, operations, and algebraic thinking may lend itself best to practice games, develop and use games that involve measurement, geometry, data, and other domains and clusters from the Core.

Example: Grade 4, Domain: Operations and algebraic thinking, Cluster: Gain familiarity with factors and multiples, Standard 4: *Find all factor pairs for a whole number in the range 1–100. Recognize that a whole number is a multiple of each of its factors. Determine whether a given whole number in the range 1–100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1–100 is prime or composite.*

2. Within the standard, identify possible practice areas. For the example above, *Finding all factor pairs for a whole number in the range 1–100* is one such area. One of my favorite game published by the NCTM is the Factor Trail game (Vennebush, 2014; http://illuminations.nctm.org/Lesson.aspx?id=2520) which requires students to find all factors of a number, to use strategy in making their decisions to move on the game board because this will affect the outcome of the game, and to calculate their final result. For a game involving pairs of factors, a similar game board containing paths with prime and composite numbers could be developed, and students' task will be to roll a die and land on a number to determine all factor pairs for the number they land. These pairs should be recorded, and partners could check for pairs their peer have missed and receive credit for the finds. The game could be varied with composite numbers only, or with numbers in the range 50-100 and so on. For variety and a different practice game, instead of a game board with numbers, the numbers could be determined by drawing from a pile of number cards 1-100 or

desirable strategy (Steinthorsdottir and Sriraman 2009).

HOW DO STUDENTS SOLVE THE PROBLEM?

In our study, the Mr. Tall and Mr. Short problem was part of a pencil-and-paper instrument in which students were instructed to explain their thinking. We coded student explanations by using two categories for erroneous strategies, illogical and additive, and two categories for correct reasoning, build-up and multiplicative. We then refined the multiplicative category to capture how students used the multiplicative structure of the proportion. **Table 1** defines these categories; **table 2** provides a breakdown of strategy by grade. In the following paragraphs, we illustrate each category with student work and discuss possible interpretations.

The illogical category is a collection of error strategies for work that

Table 1 Student explanations are coded using categories for erroneous and correct reasoning.

Types of Student Reasoning		
Illogical	No explanation is given; guesses and random computations are used.	
Additive	The difference between two of the quantities (either in the same or different measure spaces) is computed and applied to the third quantity. The comparisons are absolute rather than relative.	
Build-Up	The given ratio of first-measure-space quantity to second-measure-space quantity is repeated and combined using addition or multiplication as a form of repeated addition.	
Multiplicative	A multiplicative relationship is explicitly used.	
	Between-Measure Space	The given between-measure-space ratio is maintained in the target ratio.
	Within-Measure Space	The scale factor is determined and applied within each measure space, a reduced rate is scaled, or a unit rate is scaled.
Ambiguous	It is impossible to distinguish whether the student is building up (using addition) or scaling down and then scaling up (using multiplication).	
Multiplicative-Ambiguous	It is impossible to distinguish whether the student is using the within- or between-measure-space ratio.	

Table 2 This table enumerates student explanations and their frequency as a percentage.

Category		Grade 5 (n = 62)	Grade 6 (n = 88)	Grade 7 (n = 107)	Grade 8 (n = 155)
Illogical (error)		48	23	23	22
Additive (error)		39	60	50	33
Build-up (correct)		6	10	11	8
Ambiguous (correct)		5	3	3	8
Multiplicative (correct)	Between	0	0	1	3
	Within	0	0	6	16
	Ambiguous	2	3	6	10

Fig. 1 The Mr. Tall and Mr. Short problem charted students' development of abstract reasoning.

In the picture, you can see the height of Mr. Short measured with paperclips. Mr. Short has a friend, Mr. Tall. When we measured their heights with matchsticks, Mr. Short's height is 4 matchsticks and Mr. Tall's height is 6 matchsticks. How many paperclips are needed for Mr. Tall?

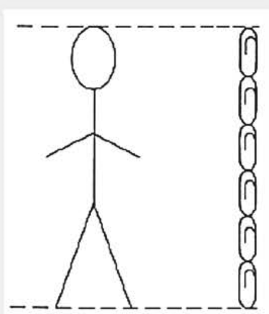
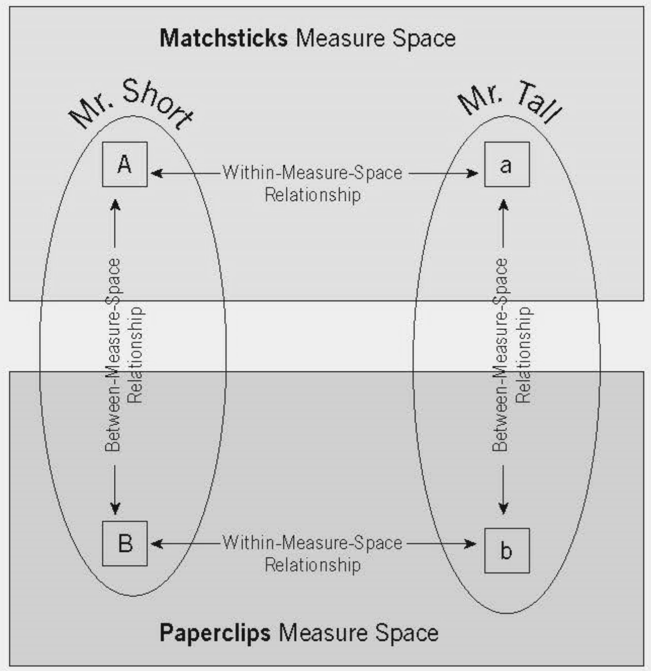


Fig. 2 The number structure can be seen by the fact that the ratio between the matchsticks is the same as the ratio between the paperclips.



to discuss what these strategies reveal about student thinking.

When teachers use the challenge with their students, our categorization may help them determine how best to nurture their students' development. The classic Mr. Tall and Mr. Short problem, unfortunately, has features that blur some levels of understanding. We therefore offer variations of the problem, which give a teacher more information to better gauge a student's reasoning.

MULTIPLICATIVE RELATIONSHIPS

Inherent to a proportion are the multiplicative relationships between the numbers involved. To distinguish among these relationships, we use the concept of a *measure space* (Vergnaud 1983). Two quantities are considered to be in the same measure space when their units are the same (see **fig. 2**). In the Mr. Tall and Mr. Short problem, for example, there is a matchsticks measure space and a paperclips measure space. We call the multiplicative relationship of Mr. Tall's height measured in matchsticks to Mr. Short's height measured in matchsticks a *within-measure-space ratio*. The multiplicative relationship of Mr. Short's height in paperclips to Mr. Short's height in matchsticks is a *between-measure-space ratio*. When we refer to the *number structure* of a proportion, we mean both the *within-* and *between-*measure-space ratios.

In a proportion, the two *within-*measure-space ratios are equal and the two *between-*measure-space ratios are equal. That is, the matchstick-to-matchstick relationship is the same as the paperclip-to-paperclip relationship ($A{:}a = B{:}b$), and the matchstick-to-paperclip relationship for Mr. Short is the same as the matchstick-to-paperclip relationship for Mr. Tall ($A{:}B = a{:}b$). (See **fig. 2**.)

Students succeed in solving proportion problems when they have at least a rudimentary understanding of one of these multiplicative relationships. When they understand both, they will flexibly choose to use whichever relationship permits an efficient solution. This more robust understanding allows students to apply and extend the power of proportional reasoning in diverse settings. As their experience with proportional situations increases, students will also be able to decontextualize and think of proportions algebraically or as equivalent fractions. Only at this point, we maintain, is an algorithmic approach of solving the resulting equation a

a selected subset. In either game, students should monitor and justify their own and their partner decisions.

3. Students should keep a record of their numbers and the factor pairs. The teacher should observe and/or collect the recording sheets for formative assessment.
4. After students finish the game (and the design of the game allows for it to be stopped and a score to be determined without reaching the end), they should share as class examples of numbers that have multiple pairs of factors and reason about them.
5. Based on students' progress in the games, the teacher determines the need for more experiences with pairs of factors. As a next step and a good candidate for another game, the teacher could think of a game-like activity for students to *Determine whether a given whole number in the range 1–100 is a multiple of a given one-digit number.*
6. As a follow up for the game on pairs of factors, the teacher may ask the class design a similar game where they include numbers with most pairs of factors. Then, they could compare games and try out their respective games.

The game format for determining factor pairs is traditionally more engaging for students than an individual paper and pencil work, and the opportunities for discourse and peer monitoring allow for continuing reasoning and a variety of connections, thus allowing the students to engage in core mathematical practices.

This process of making decisions about game use in the classroom should also be used when selecting existing, published games as well as when designing new activities. Concrete materials, like manipulatives and real life objects, could also be used, especially with young learners in games on sorting, counting, sequencing, and addition and subtraction practice. Other supporting materials – for example, number lines, five and ten frames, place value mats, and multiplication and addition charts – could also be incorporated in games. Sets of number cards would be a versatile tool for a variety of number and operation games. They could help also with randomization of the numbers involved in a game task, which would more closely resemble a situation of encountering numbers in real life rather than working with numbers in one single pre-determined problem.

Meeting instructional objectives should be in the center of any game implementation. Student learning and advancement of understanding plus engagement in the mathematical practices should guide teacher's decision. Games and game-like activities could have a solid presence in the learning and practice of mathematical ideas, and by grounding them in the instructional objectives, we can ensure that they could provide benefits for students.

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Short

While students are solving a proportion problem, their work in a measure space will enable teachers to take the measure of their thinking.

Suzanne M. Riehl and Olof Bjorg Steinthorsdottir

Ratio, rate, and proportion are central ideas in the Common Core State Standards (CCSS) for middle-grades mathematics (CCSSI 2010). These ideas closely connect to themes in earlier grades (pattern building, multiplicative reasoning, rational number concepts) and are the foundation for understanding linear functions as well as many high school mathematics and science topics. Students develop proportional reasoning slowly, and they need many experiences in diverse contexts to build their conceptual understanding (Lamon 1995). As students journey toward mature proportional reasoning, teachers can gain insight into their thinking by carefully analyzing their solution strategies on a single problem.

Robert Karplus and his colleagues began using the Mr. Tall and Mr. Short problem (see **fig. 1**) in this way in the late 1960s. Karplus, Karplus, and Wollman (1974), influenced by Piaget, aimed to chart the development of abstract reasoning in young students. The Mr. Tall and Mr. Short task, unlike the original Piagetian tasks, did not require an understanding of physical principles and so was accessible to younger children.

Beyond its use in formal research, this problem is a "classroom challenge" explored in the 2002 NCTM Yearbook, *Making Sense of Fractions, Ratios, and Proportions* (Khoury 2002). Teachers using this challenge are encouraged to assess their students at one of four broad levels of proportional thinking. In this spirit, and as part of a larger project to examine proportional reasoning, we gave the Mr. Tall and Mr. Short problem to over 400 middle school students in a small Midwestern town. Our aim in this article is to share the categories of solution strategies we found and

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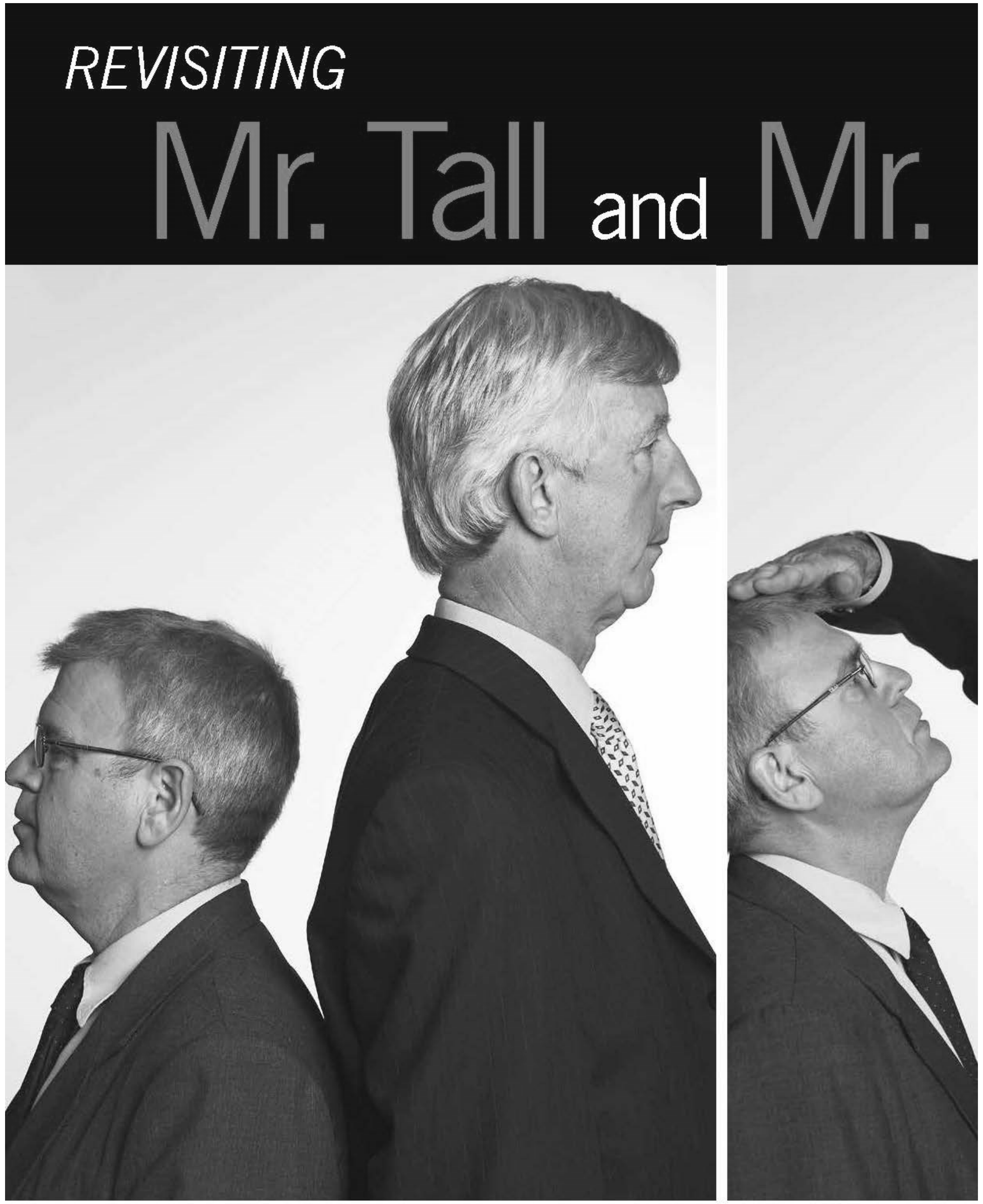

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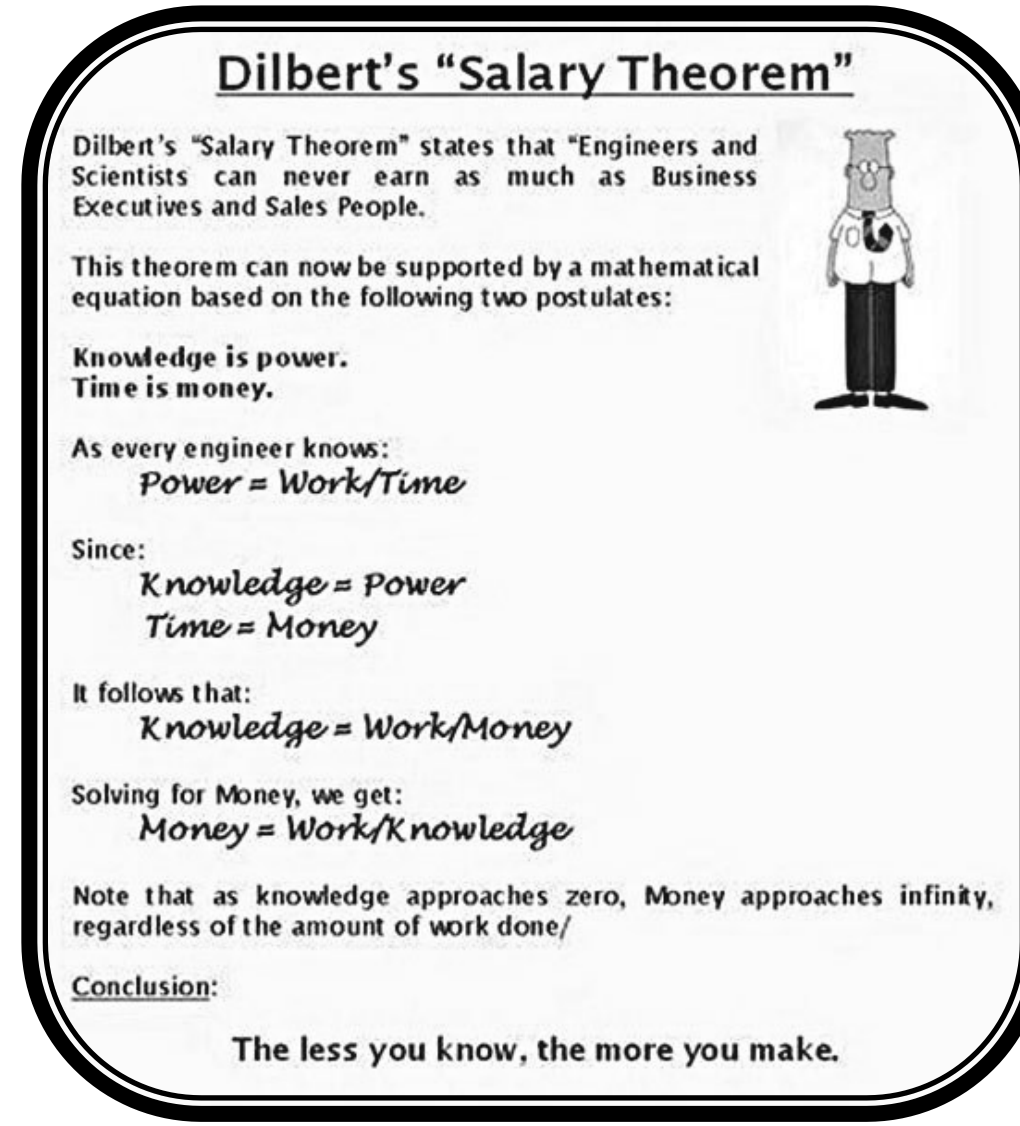
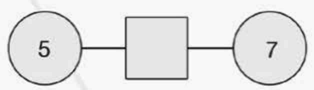


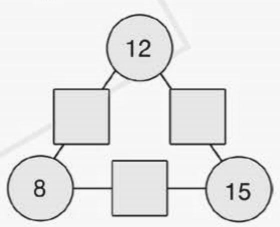
Lisa Englard

Arithmogons

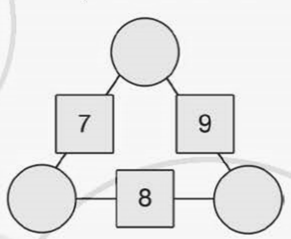
Arithmogons are easy-to-design puzzles that encourage mathematical reasoning and promote numerical fluency. One simple rule provides the basis for arithmogon puzzles: Add the numbers in the two circles to get the number in the square.



To help students become comfortable with the rule, assign triangular puzzles with numbers in circles at the vertices and empty squares to fill along the sides.



Once students understand the idea, assign triangular arithmogon puzzles with numbers in the squares and empty circles at the vertices. Challenge students to find numbers that simultaneously work for the three circles.



Arithmogons provide opportunities to investigate number patterns without the use of symbols. Students can explore odd and even numbers, fractions and mixed numbers, negative numbers, and the existence of solutions.

The following questions may stimulate class discussions about arithmogons:

- What strategies did you use to find the values at the vertices?
- Is it always possible to find the numbers at the vertices when you are given any three numbers along the sides?
- Could there be more than one solution to the puzzle?
- Could subtraction or division be used to create puzzles? If so, how?

Arithmogons can be easily adapted for different grade levels by choosing appropriate values. Students could create their own puzzles and could challenge a classmate to find a solution. Additionally, in the upper grades, multiplication could be used instead of addition. For pre-K–grade 1, consider providing "arithmogon mats" where children could create their own equations by placing counters or other objects in the circles and squares.

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Lisa Englard, lisa@khanacademy.org, is a content specialist at Khan Academy. Her work involves ensuring that the math exercises on khanacademy.org meet the focus, coherence, and rigor requirements of the Common Core Standards for Mathematics. Edited by Martha Hildebrandt, mhildebrandt@chatham.edu, who teaches undergraduate and graduate math education and mathematics courses at Chatham University in Pittsburgh, Pennsylvania. Submit your quick game, puzzle, activity, or instructional strategy along with suggestions for how teachers of different grade bands (K–grade 2, 3–4, 5–6) can use the idea. Send submissions of no more than 250 words to this department by accessing tcm.msubmit.net. See detailed submission guidelines for all departments at www.nctm.org/tcmdepartments.

What is the Mathematics Vision Project?

"What students learn is fundamentally connected with how they learn it." Deborah Ball

"Effective teaching involves observing students, listening carefully to their ideas and explanations, having mathematical goals, and using the information to make instructional decisions." NCTM

The **MVP team** has created all of the resources you need to guide students to develop conceptual understanding with "proficiency, along with factual knowledge and procedural facility."[1]

The ***classroom experience*** is composed of modules that are aligned with the Common Core State Standards for Mathematics. Each lesson begins with a worthwhile task that has been designed to develop mathematical understanding, solidify that understanding, or allow for practice of the new concepts, while focusing on the mathematical goals of the chosen learning cycle. (CMI Framework) The **MVP *classroom experience*** does not look like the traditional mathematics classroom. In the **MVP**classroom the teacher launches a rich task and then through "teacher moves" encourages students to explore, question, ponder, discuss their ideas and listen to the ideas of their classmates. In this way, the teacher connects the ***Eight Mathematical Practices***to the content.

The ***Ready, Set, Go!*** homework assignments have been correlated to the daily ***classroom experience*** and should be assigned at the close of each class session. The homework is organized into three parts. The ***Ready*** section is to help the student get ***Ready***for the upcoming work and prepare to learn new material. The ***Set*** section is for practicing the skills that are being developed in the current lessons. As students practice, the new mathematical skills become more ***Set*** or fluent. The last section of homework, called ***Go!,*** is to help students remember the skills and procedures that they have learned previously. As students mature mathematically, there are many math problems they should be able to do whenever they encounter them. The procedures for solving them become automatic. Students should be able to take off and ***Go!*** with them.

Students who need additional help with the ***Ready, Set, Go!*** assignments, can search the topic for the problem set in a popular search engine or follow the internet links, when available. Most search engines return quality resources in a reliable fashion. Most of the video links provided will take the student to specific lessons in Khan Academy. When video lessons for the problems in ***Ready, Set, Go*** were not available at http://www.khanacademy.com, applicable links from http://www.youtube.com have been included.

Professional Development Resources have been compiled into a resource page. If they are accessible electronically, the link has been given. The NCTM articles and publications require membership. A professional teacher of mathematics should be a member.

Professional Development is available from the MVP team. Our professional development comes in several forms. Patrons can attend our summer conference or an MVP team of trainers can visit your school or district for training that is catered to your specific needs. It is also possible to get a glimpse of the MVP vision of teaching and learning by attending our sessions at the NCSM or NCTM conferences.

[1] Principles and Standards for School Mathematics. NCTM, 2000.pg. 20

several minutes to discuss the question with their partners. This strategy has helped improve discussions more than any others that I have adopted.

2. If students or groups cannot answer a question or contribute to the discussion in a positive way, they must ask a question of the class. I explain that it is all right to be confused, but students are responsible for asking questions that might help them understand.

3. Always require students to ask a question when they need help. When a student says, "I don't get it," he or she may really be saying, "Show me an easy way to do this so I don't have to think." Initially, getting students to ask a question is a big improvement over "I don't get it." Students soon realize that my standards require them to think about the problem in enough depth to ask a question.

4. Require several responses to the same question. Never accept only one response to a question. Always ask for other comments, additions, clarifications, solutions, or methods. This request is difficult for students at first because they have been conditioned to believe that only one answer is correct and that only one correct way is possible to solve a problem. I explain that for them to become better thinkers, they need to investigate the many possible ways of thinking about a problem. Even if two students use the same method to solve a problem, they rarely explain their thinking in exactly the same way. Multiple explanations help other students understand and clarify their thinking. One goal is to create a student-centered classroom in which students are responsible for the conversation. To accomplish this goal, I try not to comment after each response. I simply pause and wait for the next student to offer comments. If the pause alone does not generate further discussion, I may ask, "Next?" or "What do you think about _______________'s idea?"

5. No one in a group is finished until everyone in the group can explain and defend the solution. This rule forces students to work together, communicate, and be responsible for the learning of everyone in the group. The learning of any one person is of little value unless it can be communicated to others, and those who would rather work on their own often need encouragement to develop valuable communication skills.

6. Use hand signals often. Using hand signals - thumbs up or thumbs down (a horizontal thumb means "I'm not sure") - accomplishes two things. First, by requiring all students to respond with hand signals, I ensure that all students are on task. Second, by observing the responses, I can find out how many students are having difficulty or do not understand. Watching students' faces as they think about how to respond is very revealing.

7. Never carry a pencil. If I carry a pencil with me or pick up a student's pencil, I am tempted to do the work for the student. Instead, I must take time to ask thought-provoking questions that will lead to understanding.

8. Avoid answering my own questions. Answering my own questions only confuses students because it requires them to guess which questions I really want them to think about, and I want them to think about all my questions. I also avoid rhetorical questions.

9. Ask questions of the whole group. As soon as I direct a question to an individual, I suggest to the rest of the students that they are no longer required to think.

10. Limit the use of group responses. Group responses lower the level of concern and allow some students to hide and not think about my questions.

11. Do not allow students to blurt out answers. A student's blurted out answer is a signal to the rest of the class to stop thinking. Students who develop this habit must realize that they are cheating other students of the right to think about the question.

Summary

LIKE MOST TEACHERS, I ENTERED THE TEACHING profession because I care about children. It is only natural for me to want them to be successful, but by merely telling them answers, doing things for them, or showing them shortcuts, I relieve students of their responsibilities and cheat them of the opportunity to make sense of the mathematics that they are learning. To help students engage in real learning, I must ask good questions, allow students to struggle, and place the responsibility for learning directly on their shoulders. I am convinced that children learn in more ways than I know how to teach. By listening to them, I not only give them the opportunity to develop deep understanding but also am able to develop true insights into what they know and how they think.

Making extensive changes in curriculum and instruction is a challenging process. Much can be learned about how children think and learn, from recent publications about learning styles, multiple intelligences, and brain research. Also, several reform curriculum projects funded by the National Science Foundation are now available from publishers. The Connected Mathematics Project, Mathematics in Context, and Math Scape, to name a few, artfully address issues of content and pedagogy.

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Never Say Anything a Kid Can Say!

STEVEN C. REINHART

AFTER EXTENSIVE PLANNING, I PRESENTED what should have been a masterpiece lesson. I worked several examples on the overhead projector, answered every student's question in great detail, and explained the concept so clearly that surely my students understood. The next day, however, it became obvious that the students were totally confused. In my early years of teaching, this situation happened all too often. Even though observations by my principal clearly pointed out that I was very good at explaining mathematics to my students, knew my subject matter well, and really seemed to be a dedicated and caring teacher, something was wrong. My students were capable of learning much more than they displayed.

Implementing Change over time

THE LOW LEVELS OF ACHIEVEMENT ~ of many students caused me to question ~ how I was teaching, and my search for a ~ better approach began. Making a commitment to change 10 percent of my if teaching each year, I began to collect and use materials and ideas gathered from supplements, workshops, professional journals, and university classes. Each year, my goal was simply to teach a single topic in a better way than I had the year before.

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Before long, I noticed that the familiar teacher-centered, direct-instruction model often did not fit well with the more in-depth problems and tasks that I was using. The information that I had gathered also suggested teaching in nontraditional ways. It was not enough to teach better mathematics; I also had to teach mathematics better. Making changes in instruction proved difficult because I had to learn to teach in ways that I had never observed or experienced, challenging many of the old teaching paradigms. As I moved from traditional methods of instruction to a more student-centered, problem-based approach, many of my students enjoyed my classes more. They really seemed to like working together, discussing and sharing their ideas and solutions to the interesting, often contextual, problems that I posed. The small changes that I implemented each year began to show results. In five years, I had almost completely changed both *what* and *how* I was teaching.

The Fundamental Flaw

AT SOME POINT DURING THIS METAMORPHOSIS, I concluded that a fundamental flaw existed in my teaching methods. When I was in front of the class demonstrating and explaining, I was learning a great deal, but many of my students were not! Eventually, I concluded that if my students were to ever really learn mathematics, *they* would have to do the explaining, and *I,* the listening. My definition of a good teacher has since changed from "one who explains things so well that students understand" to "one who gets students to explain things so well that they can be understood."

Getting middle school students to explain their thinking and become actively involved in classroom discussions can be a challenge. By nature, these students are self-conscious and insecure. This insecurity and the effects of negative peer pressure tend to discourage involvement. To get beyond these and other roadblocks, I have learned to ask the best possible questions and to apply strategies that require all students to participate. Adopting the goals and implementing the strategies and questioning techniques that follow have helped me develop and improve my questioning skills. At the same time, these goals and strategies help me create a classroom atmosphere in which students are actively engaged in learning mathematics and feel comfortable in sharing and discussing ideas, asking questions, and taking risks.

Questioning Strategies That Work for Me

ALTHOUGH GOOD TEACHERS PLAN DETAILED lessons that focus on the mathematical content, few take the time to plan to use specific questioning techniques on a regular basis. Improving questioning skills is difficult and takes time, practice, and planning. Strategies that work once will work again and again. Making a list of good ideas and strategies that work, revisiting the list regularly, and planning to practice selected techniques in daily lessons will make a difference.

Create a plan.
The following is a list of reminders that I have accumulated from the many outstanding teachers with whom I have worked over several years. I revisit this list often. None of these ideas is new, and I can claim none, except the first one, as my own. Although implementing any single suggestion from this list may not result in major change, used together, these suggestions can help transform a classroom. Attempting to change too much too fast may result in frustration and failure. Changing a little at a time by selecting, practicing, and refining one or two strategies or skills before moving on to others can result in continual, incremental growth. Implementing one or two techniques at a time also makes it easier for students to accept and adjust to the new expectations and standards being established.

1. Never say anything a kid can say! This one goal keeps me focused. Although I do not think that I have ever met this goal completely in anyone day or even in a given class period, it has forced me to develop and improve my questioning skills. It also sends a message to students that their participation is essential. Every time I am tempted to tell students something, I try to ask a question instead.

2. Ask good questions. Good questions require more than recalling a fact or reproducing a skill. By asking good questions, I encourage students to think about, and reflect on, the mathematics they are learning. A student should be able to learn from answering my question, and I should be able to learn something about what the student knows or does not know from her or his response. Quite simply, I ask good questions to get students to think and to inform me about what they know. The best questions are open ended, those for which more than one way to solve the problem or more than one acceptable response may be possible.

3. Use more process questions than product questions. Product questions-those that require short answers or a yes or no response or those that rely almost completely on memory-provide little information about what a student knows. To find out what a student understands, I ask process questions that require the student to reflect, analyze, and explain his or her thinking and reasoning. Process questions require students to think at much higher levels.

4. Replace lectures with sets of questions. When tempted to present information in the form of a lecture, I remind myself of this definition of a lecture: 'The transfer of information from the notes of the lecturer to the notes of the student without passing through the minds of either." If I am still tempted, I ask myself the humbling question "What percent of my students will actually be listening to me?"

5. Be patient. Wait time is very important. Although some students always seem to have their hands raised immediately, most need more time to process their thoughts. If I always call on one of the first students who volunteers, I am cheating those who need more time to think about, and process a response to, my question. Even very capable students can begin to doubt their abilities, and many eventually stop thinking about my questions altogether. Increasing wait time to five seconds or longer can result in more and better responses.

Good discussions take time; at first, I was uncomfortable in taking so much time to discuss a single question or problem. The urge to simply tell my students and move on for the sake of expedience was considerable. Eventually, I began to see the value in what I now refer to as a "less is more" philosophy. I now believe that all students learn more when I pose a high-quality problem and give them the necessary time to investigate, process their thoughts, and reflect on and defend their findings.

Share with students reasons for asking questions. Students should understand that all their statements are valuable to me, even if they are incorrect or show misconceptions. I explain that I ask them questions because I am continuously evaluating what the class knows or does not know. Their comments help me make decisions and. plan the next activities.

Teach for success. If students are to value my questions and be involved in discussions, I cannot use questions to embarrass or punish. Such questions accomplish little and can make it more difficult to create an atmosphere in which students feel comfortable sharing ideas and taking risks. If a student is struggling to respond, I move on to another student quickly. As I listen to student conversations and observe their work, I also identify those who have good ideas or comments to share. Asking a shy, quiet student a question when I know that he or she has a good response is a great strategy for building confidence and self-esteem. Frequently, I alert the student ahead of time: 'That's a great idea. I'd really like you to share that with the class in a few minutes."

Be nonjudgmental about a response or comment. This goal is indispensable in encouraging discourse. Imagine being in a classroom where the teacher makes this comment: "WOW! Brittni, that was a terrific, insightful response! Who's next?" Not many middle school students have the confidence to follow a response that has been praised so highly by a teacher. If a student's response reveals a misconception and the teacher replies in a negative way, the student may be discouraged from volunteering again. Instead, encourage more discussion and move on to the next comment. Often, students disagree with one another, discover their own errors, and correct their thinking. Allowing students to listen to fellow classmates is a far more positive way to deal with misconceptions than announcing to the class that an answer is incorrect. If several students remain confused, I might say, "I'm hearing that we do not agree on this issue. Your comments and ideas have given me an idea for an activity that will help you clarify your thinking." I then plan to revisit the concept with another activity as soon as possible.

Try not to repeat students' answers. If students are to listen to one another and value one another's input, I cannot repeat or try to improve on what they say. If students realize that I will repeat or clarify what another student says, they no longer have a reason to listen. I must be patient and let students clarify their own thinking and encourage them to speak to their classmates, not just to me.

All students can speak louder - I have heard them in the halls! Yet I must be careful not to embarrass someone with a quiet voice. Because students know that I never accept just one response, they think nothing of my asking another student to paraphrase the soft-spoken comments of a classmate.

Is this the right answer?" Students frequently ask this question. My usual response to this question might be that "I'm not sure. Can you explain your thinking to me?" As soon as I tell a student that the answer is correct, thinking stops. If students explain their thinking clearly, I ask a "What if?" question to encourage them to extend their thinking.

Participation is not optional! I remind my students of this expectation regularly. Whether working in small groups or discussing a problem with the whole class, each student is expected to contribute his or her fair share. Because reminding students of this expectation is not enough, I also regularly apply several of the following techniques:

1. Use the think-pair-share strategy. Whole-group discussions are usually improved by using this technique. When I pose a new problem; present a new project, task, or activity; or simply ask a question, all students must think and work independently first. In the past, letting students begin working together on a task always allowed a few students to sit back while others took over. Requiring students to work alone first reduces this problem by placing the responsibility for learning on each student. This independent work time may vary from a few minutes to the entire class period, depending on the task.

After students have had adequate time to work independently, they are paired with partners or join small groups. In these groups, each student is required to report his or her findings or summarize his or her solution process. When teams have had the chance to share their thoughts in small groups, we come together as a class to share our findings. I do not call for volunteers but simply ask one student to report on a significant point discussed in the group. I might say, "Tanya, will you share with the class one important discovery your group made?" or "James, please summarize for us what Adam shared with you." Students generally feel much more confident in stating ideas when the responsibility for the response is being shared with a partner or group. Using the think-pair-share strategy helps me send the message that participation is not optional.

A modified version of this strategy also works in whole-group discussions. If I do not get the responses that I expect, either in quantity or quality, I give students a chance to discuss the question in small groups. On the basis of the difficulty of the question, they may have as little as fifteen seconds or as long as