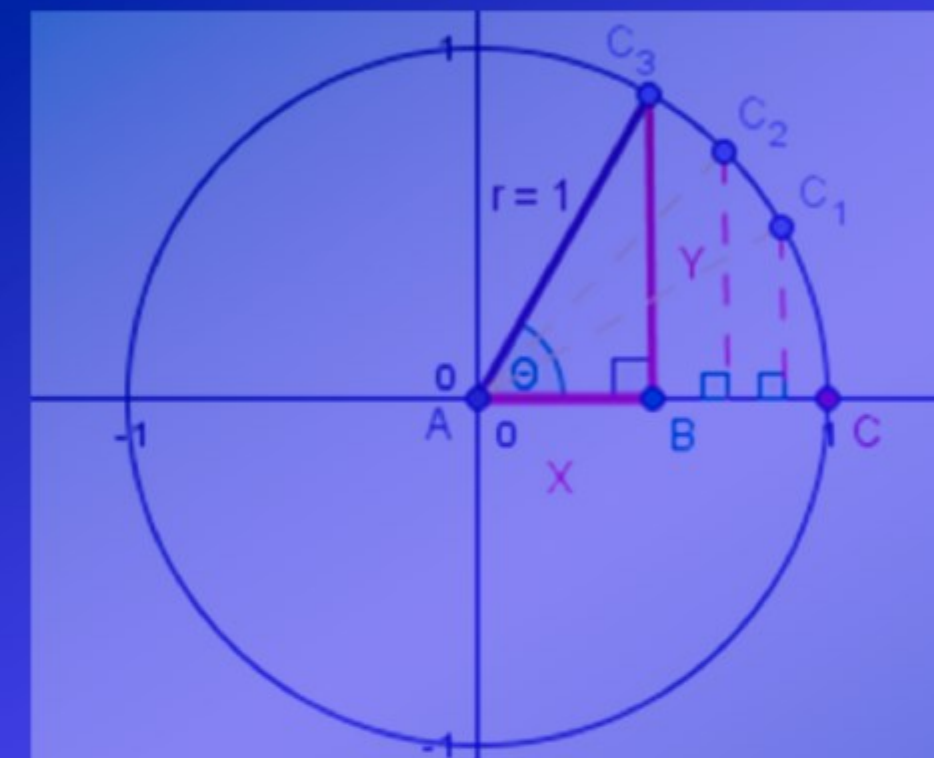
Utah Mathematics Teacher
Fall/Winter 2016-2017
Volume 9



Equity Through Practice

Utah Council of Teachers of Mathematics
UCTM

http:/utahctm.org

Call for Articles

The ***Utah Mathematics Teacher*** seeks articles on issues of interest to mathematics educators, especially K-12 classroom teachers in Utah. All are encouraged to contribute articles and opinions for any section of the journal. Some of the features are: UCTM Leader Spotlight; Letter from the NCTM President; Letter from the UCTM President; Professional Development, Puzzle Corner; Recommended Readings and Resources; Utah Core Standards and Implementation; College and University Research; and others.

Teachers are especially encouraged to submit articles including inspirational stories, exemplary lessons, beginning teacher ideas; or managements tools. Manuscripts, including tables and figures, should be typed in Microsoft Word and submitted electronically as an e-mail attachment to Christine Walker (Christine.Walker@uvu.edu). A cover letter containing author's name, address, phone, e-mail address and the article's intended audience should be included.

2015 Presidential Award for Excellence in Secondary Mathematics Teaching

Vicki Lyons is in her 23rd year teaching mathematics, 17 of which have been at Lone Peak High School where she currently teaches Honors III Mathematics, Advanced Placement (AP) Calculus BC, and AP Statistics. Her classroom experience ranges from Algebra I through college mathematics courses.

By emphasizing sociomathematical norms in her classroom, Vicki guides her students to become confident, knowledgeable, and authentic users of mathematics and statistics. Her focus is helping students develop strong logical foundations through conceptual understanding, strategic reasoning, and practiced skill.

Vicki has spoken at numerous conferences and workshops and served on many committees at the local, state, and national levels. She enjoys her present work with the Utah State Office of Education's Secondary Math Leadership Team. Amongst her many awards, Vicki recently received the Distinguished Mathematics Educator Award from Brigham Young University. Vicki serves on staff of the Teacher Leadership Program for the Institute for Advanced Study's Park City Mathematics Institute.

Vicki earned a B.S., magna cum laude, in mathematics and a M.A. in mathematics education with a minor in statistics from Brigham Young University. She is also a third year doctoral student in curriculum and instruction at Utah State University. Vicki is National Board Certified in mathematics/adolescence and young adulthood.

2014 Presidential Award for Excellence in Elementary Mathematics Teaching

Jalyn Kelley has been a professional educator in the Logan City School District for 16 years and has taught at Wilson Elementary School for the past four years. She started her career teaching kindergarten. She has taught fourth grade for the last 11 years.

Jalyn is continually looking for ways to increase her knowledge of mathematics and improve classroom instruction. Her use of purposeful depth of knowledge questioning guides student thinking and engages learners in mathematical discourse. Math Talk is a critical thinking tool that is evident in her classroom.

Jalyn is a district mathematics trainer and has provided professional development for other school districts in Utah. As a result of collaborating with Jessica Shumway, the author of "Number Sense Routines," Jalyn and her students were selected to be videotaped by Stenhouse Publishers as they worked through their number sense routines. The video, "Go Figure!" was released in May of 2014. Jalyn has presented at Utah Council of Teachers of Mathematics and the National Council of Teachers of Mathematics Conferences.

Jalyn has a B.S. in multidisciplinary studies from Eastern Oregon University and is in the process of completing her M.Ed. at Utah State University. She has completed the Utah Elementary Mathematics Endorsement and is certified to teach kindergarten through eighth grade.

2016 Presidential Awardees for Excellence in Elementary Mathematics Teaching State Semi-Finalists

Julie Christensen currently teaches 3rd grade at Neil Armstrong Academy in Granite School District. She is in her 11th year of teaching. In addition she has taught 1st grade, 3rd grade, and 5th grade, as well as a school math specialist and a school lead ESL teacher. She received her undergraduate degree from Dixie State College, and a Masters of Education from Southern Utah University. She also has an ESL endorsement, math endorsement, technology endorsement and reading endorsement. Although she loves teaching all subjects, her true passion is mathematics.

She resides in Stansbury Park, Utah with her husband and two children. In her free time, she loves spending time with her family and being crafty!

Jennifer M Bodell has been an educator for 20 years in the Granite School District. She has taught 5th and 6th grade at Neil Armstrong Academy and Truman Elementary. She currently is the Instructional Coach at the new West Lake STEM Junior High. She was Granite School District's Teacher of the Year in 2012 and Utah Teacher of the Year runner up. Math was a subject that was very difficult for Jennifer during her own elementary years, and she vowed when becoming a teacher she would teach math in a way that all types of learners would understand and love. One of the best ways she believes doing this is creating real-life math situations and investigations allowing students to make connections and discoveries to provide them their own "aha" experiences. Jennifer also is an advocate of STEM and knows the importance of integrating subjects together to help students be more prepared for the future, especially in the STEM fields.

Utah Mathematics Teacher

Table of Contents

Fall/Winter 2016-2017 **Volume 9**

UCTM President Message

Joleigh Honey, Coordinator
Science, Technology, Engineering, Mathematics (STEM), Secondary Mathematics (7-12)

We've come a long way!

It is my privilege to introduce the ninth volume of the *Utah Mathematics Teacher.* Launched in 2008, the journal reflects the depth and diversity of thought and academic excellence that Utah teachers strive to achieve.

It is in this message that I salute the mathematics education community and to share the growth of mathematics education over the past eight years. We still have a ways to go, yet we also have much to celebrate! We have indeed come a long way. Over the past eight years, we have more students graduating from high school, fewer students repeating courses, more students taking advanced mathematics courses, and we are doing our part to close the opportunity gap. Here are some examples of how far we have come:

- Over the past five years, enrollment in AP courses and IB mathematics courses have increased:

Students Enrolled in AP or IB Mathematics Courses (11th and 12th graders)			
Year	Number of Students Enrolled	Total Number of Students	% of Students Enrolled
2011	7081	78039	9.07%
2012	7235	80183	9.02%
2013	8046	81568	9.86%
2014	8269	83077	9.95%
2015	8822	85634	10.30%

From transitioning from the 2007 Core to the 2010 Core, we have increased expectations, instruction, and are (mostly) getting the support we need to help students succeed!

Increasing the number and percent of students staying "On Grade Level"

- In 2008, only 73% of students completed Algebra I by 9th grade.
- In 2016, more than 87% completed Secondary I (a more rigorous course) by 9th grade.

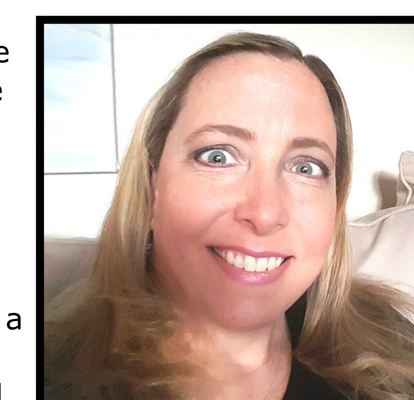
Increasing the number and percent of Special Education students taking and succeeding in more rigorous courses:

- In 2008, there were 5,431 Special Education students taking Algebra I, with only 43% of these students taking the course on or before grade level (as 7th, 8th, or 9th graders).

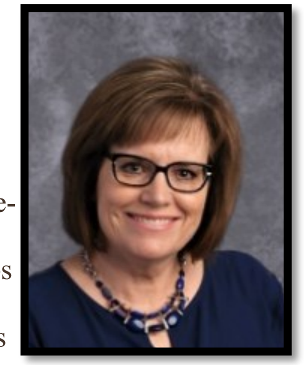
2016 Presidential Awardees for Excellence in Elementary Mathematics Teaching State Semi-Finalists

Nancy Stewart currently teaches 3rd grade in Logan Utah at Edith Bowen Laboratory School on the USU campus. She has taught for 21 years and has her ESL, Reading, and Math endorsement. She loves teaching and wants to help children learn to think deeply, value themselves, and empower them to reach their potential. She is extremely interested in math education and has been a district math trainer, has presented at national math conventions and at UCTM. She values teaching at a laboratory school as it gives her the opportunity to continually mentor future teachers, collaborate with USU professors, and try out new methods. She believes in facilitating mathematical UNDERSTANDING and number sense. She loves being with her family, reading, spending time at her Bear Lake family cabin, and traveling.

Carrie L. Caldwell was born and raised in Ohio. Upon completing her education at the University of Kentucky, she moved to North Carolina and taught fourth and fifth grade for 5 years. She then moved back to the Northern Kentucky area and taught in an inner city school outside of Cincinnati. During this time, she earned her Master's in Urban Educational Leadership from the University of Cincinnati. In 2009, she accepted a position as an Instructional Math Coach with the Salt Lake City School District and moved to Utah. She served as a coach for 4 years before the pull of the classroom took her back. She has taught first grade at Riley Elementary in SLCSD for 3 years and currently teaches fourth grade. In addition, she teaches the Elementary Math Methods course at the University of Utah. She has an intense passion for not only teaching children mathematics, but adults and preservice teachers about mathematics education. She loves watching her students make mistakes and then discover the mathematics, and indicates that it's quite amazing to see. If she is not in a classroom you usually can find her on a mountain with her lab Moose or her part-time job at Snowbird.



Karl Jones—Jan Farmer



Edith Bowen Laboratory School at Utah State University

Parents of Mrs. Farmer indicate that she is an outstanding and enthusiastic mathematics teacher. In particular, one parent remarked that she has been amazed by her daughter's attitude and growth in mathematics proficiency. Mrs. Farmer helps students develop goals appropriate for their grade level - but maintains a very high standard about what that means. In addition, Mrs. Farmer helps her students develop a deep understanding of the decimal place value system and make sense of complex algorithms. She also helps her students gain proficiency with multiplication facts and yet at the same time develop a sense that mathematics is about reasoning and justification. Parents are thrilled with the work that Mrs. Farmer has done to lead her students to high levels of achievement and understanding.

George Shell—Roger Haglund

West High in Salt Lake City School District

Mr. Haglund has achieved superior results and created positive differentiation from others within his field through innovation and creativity in approaches, techniques, methods and processes. Using data, Roger Haglund developed his own innovative data templates to hold himself accountable for his students' learning. He utilized data to inform his instruction to meet his students' needs and has shared these data templates within his school and throughout the school district in order to help other teachers improve their students' learning. These data templates helped the Northwest Middle School math department focus their instruction to target specific math concepts on which students struggled. As a result, the math department has received the middle school academic achievement award for student growth on the CRTs for the last two years. Recently, national attention was given to the school by Arne Duncan, the secretary of education, as he observed Roger Haglund teaching a math lesson at Northwest Middle. This was in part due to Mr. Haglund's student's scores and academic performance. One of the reasons for his success as an educator is that he always makes his lessons interesting for his students and their lives. He creates many of his own lessons and shares them with his department. He brings learning to life with technology specifically using the TI Nspire calculator. Students regularly work on excel documents to represent data they collect. In one such lesson, Mr. Haglund has his students build bridges using excel spreadsheets to graph the data and make predictions. Mr. Haglund has also found that using flip cameras and Ipod's instrumental in teaching EL (English Learners) students. He does this by forming small groups of students to teach each other the math concepts. Student's then video record themselves explaining math to one another. Many shy students and students new to the English language are willing to speak about math into the video even though they will not respond in class. This helps Mr. Haglund check for student's understanding. It also helps all students' solidify the math concepts and practice math in a fun way using math games for extra practice, which leads to high motivation using technology!

- In 2016, more than 87% completed Secondary I (a more rigorous course) by 9th grade.

Increasing the number and percent of Special Education students taking and succeeding in more rigorous courses:

- In 2008, there were 5,431 Special Education students taking Algebra I, with only 43% of these students taking the course on or before grade level (as 7th, 8th, or 9th graders).
- In 2016, there were 7,267 Special Education students taking Secondary I, with more than 65% of these students taking the course as 7th, 8th, or 9th graders.

We should be very proud as a mathematics community!

THANK YOU for all you do- your job as an educator is the most important job one can have for the future of our state and our country. When people ask about the standards, be proud of who you are and what you do. When you are asked about the standards, explain how they are making a difference for our students and share that the Utah Council of Teachers of Mathematics has always supported the standards.

As a Board, UCTM has sent letters of support of the adoption of the Utah Core Standards of Mathematics to the legislature for several years. This past year, we have also included an advocacy statement in support of the standards. The statement is in this journal as well as on the UCTM website. We have also added advocacy as a role on the UCTM Board in an effort to provide you with support of the work we do. In the future, we will add advocacy statements similar to those from other mathematics organizations, such as those put out by the National Council of Teachers of Mathematics (NCTM). As a Board, we would like your input and also encourage you to join us. If you have data, research, and an interest, please share. You may even wish to run for a position on the Board as there are many roles where you can make a difference.

The publication of this journal is the culmination of a six month editorial process, carried out by the Journal Staff of the UCTM Board: Christine Walker (Journal Editor), Emina Alibegovic, and Jennifer Throndsen. As the President of UCTM, I am honored to work with talented and dedicated Board members who work hard every year to support and represent the mathematics education community. I also thank you, the members of UCTM, for all you do to support and promote student achievement and mathematics education in general.

Sincerely,

Joleigh Honey

Letter from the Editor

Christine Walker, Utah Valley University

The theme of the Fall/Winter 2016-2017 *Utah Mathematics Teacher* Journal and the Fall 2016 UCTM Conference is "Equity Through Practice." Many ask what does equity through practice in mathematics education mean? In a National Council of Teachers of Mathematics position statement, equity through practice is characterized by;

> Practices that support access and equity require comprehensive understanding. These practices include, but are not limited to, holding high expectations, ensuring access to high-quality mathematics curriculum and instruction, allowing adequate time for students to learn, placing appropriate emphasis on differentiated processes that broaden students' productive engagement with mathematics, and making strategic use of human and material resources. When access and equity have been successfully addressed, student outcomes—including achievement on a range of mathematics assessments, disposition toward mathematics, and persistence in the mathematics pipeline—transcend, and cannot be predicted by students' racial, ethnic, linguistic, gender, and socioeconomic backgrounds.
> Access and Equity in Mathematics Education, NCTM, www.nctm.org/Standards-and-Positions

In an era of globalization, our students today need to learn to think analytically and creatively, interact with colleagues from different cultural backgrounds, and be flexible in mathematical thinking as it relates to other disciplines.

In this issue you will find articles that highlight innovative work, written by skilled and effective teachers who promote high-quality educational opportunities imperative for realizing the vision of "equity through practice," and we begin with a message from our NCTM President Matt Larson, *A Renewed Focus on Access, Equity, and Empowerment.*

One of the goals as outlined by NCTM regarding a mathematics classroom is that students should treat each other with respect, and value the contributions of others as they interact with their colleagues. In the article *Making Magic: Lessons from Improv*, pay special attention to the authors use of "etiquette of the group mind" which focuses on students learning how to "alter their conceptions after listening to each other."

We then turn our attention to two articles that focus on high-quality mathematics curriculum and instruction that support access and equity. *Improving Elementary Students' and Preservice Elementary Teachers' Attitudes and Knowledge Related to STEM Subjects through an Enrichment Robotics Program* and *Teaching Trigonometry: Are we missing the point?*, where the authors encourages teachers to couch trigonometry within a historical perspective to change and enrich the way students interact with mathematics as a way to promote equity in mathematics instruction.

In keeping with the theme of "equity through practice," NCTM recommends that we consider a "range of mathematics assessment" for equity to be successfully addressed. *A Review of Literature: Assessment Literacy for Mathematics Teachers and Students* takes a close look at what it means to have assessment literacy in order to "maximize the benefits of using data gathered from formative assessments to adjust instruction and study habits."

We close with two articles, *A Rational Approach to Irrational Numbers*, and *A Brief History of τ: A Useful Alternative to π,* that utilize inquiry and historical context to "broaden students' productive engagement with mathematics." Matthew Felton, co-author of *Connecting the NCTM Process Standards and the CCSSM Practices* wrote that for students to make connections across representations and mathematical concepts; students should study mathematics by learning about "human thinking and accomplishments throughout history."

We hope you enjoy this journal and as always, please consider submitting your own articles, or serving as a reviewer for future journal articles.

In conclusion, a very sincere thanks goes to Emina Alibegovic and Jennifer Throndsen for their work as co-editors of the *Utah Mathematics Teacher* for the Fall/Winter 2016-2017 issue. They both made significant contributions to the articles that are included in this journal.

Note: Any mistakes are the sole responsibility of the editor and will be remedied in the online journal. Please reference the online journal for full color images and live links.

Muffet Reeves—Dixie Blackington

Weber State University

Dixie Blackington has taught mathematics and mathematics education courses at Weber State University for 30 years. Each semester she tries new strategies to help her student's gain a deeper understanding of mathematics. Even though she plans to retire at the end of this year, she is still reworking her courses in an attempt to better prepare future teachers of mathematics. Her teaching reaches beyond Weber State to the local districts where she has taught elementary mathematics specialist and endorsement courses for years, and provided mathematics professional development for high school, junior high and elementary teachers. Her dedication to teaching and learning is unparalleled and her students applaud her for this. She is beloved by hundreds of area elementary teachers who speak fondly of being pushed to understand mathematics in ways they never had before having her as a teacher. Dixie Blackington, and her long-time colleague Diane Pugmire, were pioneers that paved the way for the Elementary Mathematics Endorsement program. Both made countless contributions to the state of mathematics education in Utah, and advocated for changes at the local and state levels. Over the years, she has served on numerous committees for mathematics education both at Weber State and across the state. Recently she served on the Governor's Technical Review Team, overseeing the technical review of the Common Core State Math Standards. In addition, she has also authored two textbooks.

Don Clark—David Smith

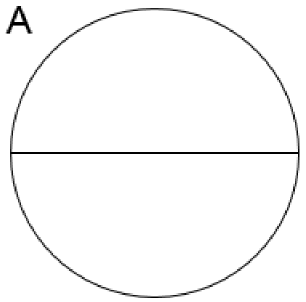
Utah State Board of Education

David Smith has committed his life to mathematical understanding as well as helping teachers deepen their own understanding. As a former elementary teacher and principal, he began his work at the Utah State Office of Education with a solid knowledge of mathematics for elementary schools, yet he set goals to deepen his knowledge by signing up for online courses, and making strong connections with Bill McCallum and others. He has brought many outstanding programs to Utah, including Jo Boaler's MOOC, Open Education Resources, and the STEM Principals Academy. He has a reputation for being supportive and appreciated within the state and by other mathematics specialists outside of the state, by playing an active in the Association of State Supervisors of Mathematics. In all the many roles that Mr. Smith has, his strength comes from his calmness and his ability to see various sides of any issue. In one particular instance, as remembered by his colleagues, he was conducting a very dicey conversation about mathematics assessment. Mathematics coordinators practically had pitchforks and cattle prods as they asked questions about upcoming assessments and the new core in 2002. He smiled at the group and calmly explained why the state was moving in the direction of assessing students on the new core and very quickly won over the support and cooperation of the math coordinators. For the past seven years David has tirelessly worked to improve mathematics education in Utah. He is known to be a constant source of optimism and support.

m&m math with CIRCLES

Use the diameter of the candy to measure the CIRCUMFERENCE and the DIAMETER of each circle.
Record how many candies it takes (you may use halves). ATTEND TO PRECISION!

A



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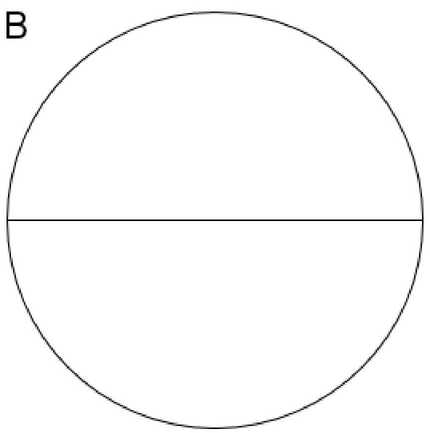

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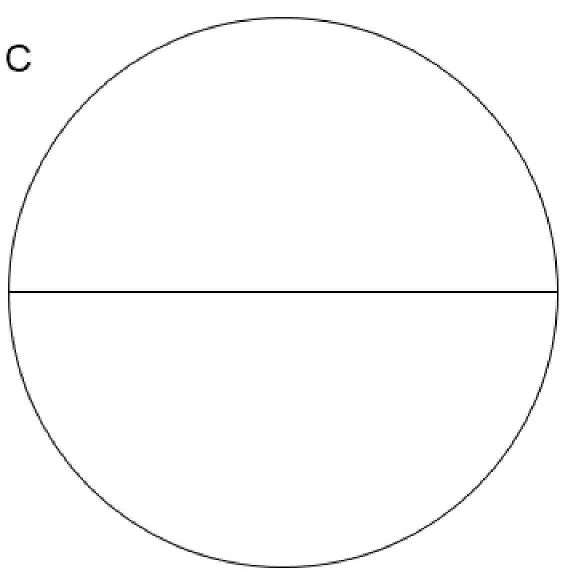

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B			
C			
		Average	

UCTM President-Elect Message

Karen Feld, Lehi Junior High

It is an exciting time to be a math educator! It seems like there is so much available to help deepen our understanding of mathematics. Our state secondary math specialist, Joleigh Honey, has released a series of courses designed specifically for Intermediate 1, Intermediate 2, and Secondary Mathematics 1. They cause teachers to look at the core in a deep and meaningful way to help make connections within and throughout the grade levels. The NCTM journals allow us to think about specific topics in each grade band and give practical lessons and teaching practices to help our students learn. We also live in an era where data collection is at our fingertips, helping us to identify a need for tier 2 and tier 3 interventions. UCTM has recently instituted a "Twitter chat" where we can get on Twitter as educators and chat about important topics, share ideas, and grow as a mathematics education community.

With all of these resources available to us, I would like to bring your focus to a few that I hope you will take the time to read, think about, and apply during this school year. The first resource comes from NCTM, which has produced three publications that have assisted in revolutionizing mathematics education over the years. In 1989, NCTM published the *NCTM standards*, which presented a vision of appropriate mathematical goals for all students. In 2000, NCTM published the *Principles and Standards for School Mathematics* (PSSM), which is a comprehensive and coherent set of mathematics standards for each and every student. And now, in 2014, NCTM has available their landmark publication, *Principles to Actions*, which connects research to practice. Included in this publication are the eight teaching practices, which, if implemented in our classrooms, will allow deeper learning and productive discourse to take precedence. I highly encourage each of you to obtain a copy of *Principles to Actions*, read it, and make an action plan for yourself. How will you use this publication to become a more effective teacher this year and in years to come?

The next resource that has been an integral part of shifting the way I think about mathematics education is Jo Boaler's book *Mathematical Mindsets* and her online course *How to Learn Math*. The book focuses on how to increase a growth mindset in our students by looking at different facets of education including homework, feedback on tests, and equity in the classroom. The online course also helped me deepen my understanding of how to teach students to have a growth mindset, how to encourage them to make and learn from their mistakes, and how to encourage deeper thinking while learning algebraic concepts. Again, I would encourage you to read this book and make an action plan for yourself. What does a mathematical mindset mean to you? What can you do to ensure high levels of learning for all students?

Both of these publications are changing the way mathematics is being taught. We want our students to be ready for any and all opportunities that will come their way in the future. To do that we need to be willing to make changes by apply the teaching practices in our classrooms, be aware of and decrease the equity issues that exist, help our students to productively struggle, and become advocates for this amazing profession we are all part of. This may require us to step out of our comfort zone, collaborate with others in our school, district, state, and nation, be willing to try something new, and know that the mistakes we make will only help our brains to grow and our teaching to improve.

I am so excited to be serving as your UCTM President for the next two years. I hope to be able to talk with you about how things are going in your classrooms, what concerns you have, what successes you have, and, most of all, how the UCTM board can help you make the changes you want to make in your classroom. I am honored to be an educator in Utah. It is a privilege to be teaching with you and learning along side you.

Karen Feld
President-Elect, UCTM

NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

A Renewed Focus on Access, Equity, And Empowerment

by NCTM President Matt Larson
September 15, 2016

In their joint position statement Mathematics Education Through the Lens of Social Justice: Acknowledgment, Actions, and Accountability, the National Council of Supervisors of Mathematics and TODOS: Mathematics for ALL identify social justice as *a key priority in the access to, engagement with, and advancement in mathematics education for our country's youth. A social justice stance requires a systemic approach that includes fair and equitable teaching practices, high expectations for all students, access to rich, rigorous, and relevant mathematics, and strong family/community relationships to promote positive mathematics learning and achievement. Equally important, a social justice stance interrogates and challenges the roles power, privilege, and oppression play in the current unjust system of mathematics education—and in society as a whole (NCSM & TODOS, 2016, p. 1).*

At its July meeting, the NCTM Board of Directors unanimously voted to endorse the NCSM/TODOS joint position statement. Challenged by Danny Martin's critique of *Principles to Actions* at the NCTM Research Conference in Boston in 2015, NCTM began, with the help of critical friends, to question and reassess its equity stance, actions, and language. NCTM also began to increase its collaborative actions concerning access, equity, and empowerment issues in a manner that embraces excellence for each and every student.

NCTM has long written about access and equity, developing, perhaps most notably, the Equity Principle in *Principles and Standards for School Mathematics* (NCTM, 2000), the NCTM position statement on *Closing the Opportunity Gap in Mathematics Education*, and, most recently, the Access and Equity Principle in *Principles to Actions: Ensuring Mathematical Success for All Students* (NCTM, 2014). Yet, our need to refine, refocus, and build on these efforts is clear. We recognize that much of our work has focused on standards, curriculum, instructional practices, and assessment, and that we have too often addressed these issues in decontextualized ways that have frequently ignored the experiences and realities of children's lives. We are committed to making

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[1] Jensen, E. (1998). Teaching with the brain in mind. Alexandria, VA: Association for Supervision and Curriculum Development.

[2] Johnson-Laird, P. N. (1983). Mental models: Toward a cognitive science of language, inference, and consciousness. Cambridge, MA: Harvard University Press.

[3] Linggou, B., & Schoen, R. (2011). *Model-Centered Learning Pathways to Mathematical Understanding Using GeoGebra* (Vol. 6). Rotterdam, Netherlands: Sense.

[4] Sirotic, N., & Zazkis, A. (2006) *Irrational Numbers: The Gap between Formal and Intuitive Knowledge*. Educational Studies in Mathematics 65.1, 49-76.

than I've ever heard. Students' eyes were amazed that we could instantly travel a million digits into π and find strings of numbers. A highlight of this piece was a student asking "when does 14159 appear again?" Her question was immediately followed by "Yeah! I want to see when it repeats!" I searched the string, and we found it, but it was followed by 5 and not 2. So we found the next one, but it was followed by 7. So we searched 141592, but quickly realized it didn't begin to repeat either. Students were amazed that it would never repeat. I jotted down during class that one student said "this hurts my head..." Other students agreed. This was my favorite comment of the week. It was also evidence that exploring π was worthwhile, as students have been continuing to come and ask me questions. As a teacher, among my goals is instilling a sense of wonderment and awe in my students. I have been on a journey of incommensurability for almost a year now, and I wish it had started sooner. Wonder is a powerful driving force in our students, and irrationality is something the world has wondered about for thousands of years. I am glad I could see students off on this mathematical adventure!

Another modification that was successful was finding a precise definition of a circle. The segue into this came from my n-gon presentation. I showed students the 128-gon and a student asked "is that what we see on a computer? Do we see a polygon with lots and lots of sides?" It was a moment where as teachers we say "I couldn't have paid you to say that!" It was so powerful to me that I went back and included it into my modified lesson plan for my own future use. It was a lesson in the difficulty of writing a precise definition. I pretended I had never seen a circle before, and they were to instruct me in drawing a circle. There were many imprecise definitions given to me, but eventually we decided between two: One student suggested that I tie a string to a pencil and anchor the string at a center point, then use the pencil attached to the string to draw a line all the way around. The other suggestion was to find infinitely many lines of the same length, and center them all at a point in all directions.

The conversation that came from the two suggestions was powerful in its context of objects of one dimension vs two dimensions. I asked students how many dimensions they thought a circle had. Most students thought a circle was two dimensions. This is a common misconception with many teachers, since in fact a circle is one dimension. Had I simply given students a definition of a circle to students and not forced them to be precise, there is a low chance that this discussion could have come up organically during class. I marveled at the connections students made to finding a true definition, and was thrilled to address a common misconception about what a shape itself is vs finding the area of a shape.

I had originally planned in my lesson a section where we graphed the relationship between the M&M's that measured the circumference and diameter of a circle. I realized partway through the lesson that students were not ready to have this discussion yet. It occurred to me that if a line were graphed with π as the constant of proportionality, the line would not pass through any lattice points (except for the origin). Our line of best fit on the approximation graph would likely pass through a lattice point, and I felt as though this would obscure the goal of this lesson. I do think this graphing activity has a lot of potential depth and is something I would like to include in future lessons, however I am unsure of how to utilize it. Your suggestions are welcome at kierstenthorsen@gmail.com.

NCTM
NATIONAL COUNCIL OF
TEACHERS OF MATHEMATICS

our actions consistent with our words to give them full power and impact.

Simply put, NCTM has traditionally concerned itself with advocating for students to have access to the mathematics. Each and every student's access to a rigorous and coherent curriculum, coupled with highly effective instruction, remains a significant challenge in the United States (see 2013–2014 Civil Rights Data Collection—First Look from the U.S. Department of Education). Significant structural obstacles, including tracking and teacher assignments that disadvantage students who have been marginalized, remain unacceptable practices in too many schools. Moving forward, NCTM pledges to devote more attention to what happens to students once they have access to rigorous mathematics courses. We need to ensure that no student is denied high-quality learning through his or her experiences in the classroom. Equity requires excellence for each and every student.

In addition to endorsing the National Council of Supervisors of Mathematics and TODOS joint position statement on mathematics education and social justice, NCTM will strive to transform its vision and actions. This initiative is moving forward on multiple fronts:

- The NCTM Board has officially reframed its equity work to focus on Access, Equity ***and*** **Empowerment** to capture the critical constructs of students' mathematical identities, sense of agency, and social justice. The Board has modified its strategic priorities to reflect this reframing of NCTM's scope to include more than just access and equity.

- Under the leadership of Diane Briars, NCTM hosted an Equity Initiative Meeting (March 2016) with representatives from some of our major Affiliates, including TODOS, NCSM, the Association of Mathematics Teacher Educators, the Association of State Supervisors of Mathematics, and the Benjamin Banneker Association. Through dialogue and conversations, participants worked to build common ground, fostering greater collaboration among these leading organizations to address these critical issues and hold one another accountable for actions moving forward.

- NCTM has joined a number of other mathematics education organizations, including NCSM, TODOS, Benjamin Banneker, and AMTE this year in participating in *A Call for A Collective Action to Develop Awareness: Equity and Social Justice in Education*.

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- NCTM has committed itself to monitoring its language to ensure that students are positioned as having assets and not deficits.
- Equity and empowerment will be included deliberately and thoughtfully in the forthcoming *Principles to Actions* Elaborations Series grade-band instructional books.
- The Elaboration book addressing the Access and Equity Principle will embrace the additional concept of empowerment by including the topics of student identity, agency, and teaching mathematics for social justice.
- Access, Equity, and Empowerment will be the focus of NCTM's 2017 Innov8conference in Las Vegas (November 15–17).
- The 2018 *Annual Perspectives in Mathematics Education* will address issues surrounding access, equity, and empowerment. The volume's working title is *Rehumanizing Mathematics Teaching and Learning for Students Who Are Latina and Black*.
- A Curriculum Resources Collaboration Center on NCTM.org is being developed with the Math Forum. On this site educators will be able to share and collaborate on lessons and instructional challenges. Educators developing and using social justice learning experiences and those addressing issues of equity, identity, and opportunity in their mathematics classrooms will have spaces to collaborate, develop ideas, and to share them with the larger community.
- The NCTM Board and staff are engaging in their own professional learning on issues surrounding access, equity, and empowerment. This includes a common book study devoted to *The Impact of Identity on K–8 Mathematics: Rethinking Equity Based Practices*. The Board and staff will also engage in professional development on the impact of microaggressions in the classroom in an effort to deepen individual understanding of these critical issues.

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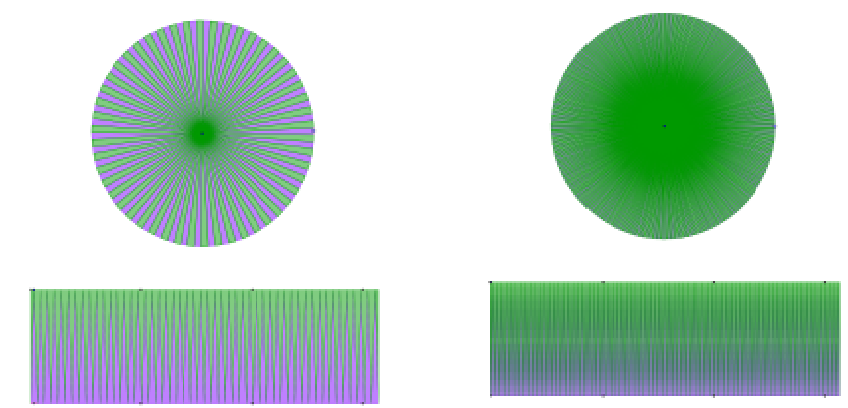
We recognize that these actions are not enough. Rather, they reflect our start and our commitment. And these are specific actions consistent with NCTM's vision statement, which states in part: "We envision a world where everyone is enthused about mathematics, sees the value and beauty of mathematics, and is **empowered** by the opportunities mathematics affords." The

The "rectangle" made of the d wedges will have a piece of patty paper over it showing that there are about 3 full copies of r^2 and then another (approximately) .14 copy of r^2 as well. The image to the right shows what is on the patty paper. Together, the area makes 3.14 copies of r^2. As the wedges are cut smaller and smaller, the base of the "rectangle" will converge to half of the circumference (πr), while the height of the rectangle will converge to r. Multiplying the two together to find area is thus πr^2.

(a) Geogebra demonstration: Wedges

Here is another technology piece so I can show students with precision what happens as our wedges become smaller and smaller. Things students should notice are: the height becomes closer and closer to being the radius, and although the edges appear to be smooth as the wedges become smaller, if we zoom in the line will still be "bumpy" no matter how many wedges we break the circle into. The computer I am using can handle 500 wedges before it becomes too difficult to zoom. Here is a visual of 100 and 500 wedges:



The rectangle shape becomes more obvious as the number of wedges increases. At 500 wedges, the image can still be zoomed in upon and there is a slight curve to the edge. This is also an excellent opportunity to break the circle into 360 wedges so students could have a visual of how small one degree of a circle is. This was part of my segue into the next topic, vertical and supplementary angles.

How Did it Go?

I was very torn in planning this lesson for my seventh graders as irrational numbers are not included in the 7th grade core. As I began planning my lesson on the circumference and area of circles, I decided that it would be a disservice to students to use π without any explanation or depth of understanding. My research outlined several reasons as to why it would be beneficial to engage in a level of exploration with irrational numbers without dedicating an entire unit to the topic. In building my new lesson, I set out to build a foundation upon which students could build their thinking in the 8th grade, when irrational numbers are formally introduced.

The entire week of circles itself was effective in that it spurred a lot of wonder from most of the students. During the piece when we found birthdays in π, the room was more silent

Unless students previously had many digits of π memorized prior to the exercise, it is likely that they cannot. I will do an activity after this with students where we find our birthdays within π, both to show the non-repeating nature of π and the vastness of the decimal expansion we have uncovered.

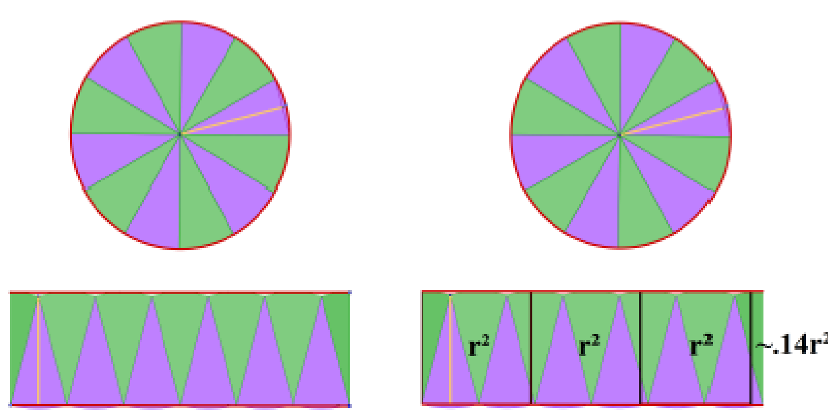
During this section I also pose the question: is our technology creating n-gons with many sides or is it creating a true circle? If so, how does a computer create a perfect circle? I will pretend to be a computer and ask students to instruct me on how to draw a circle. I want them to be thinking about the true definition of a circle, and the precision of the instructions they give.

3. Definition of a Circle

An overarching goal in all lessons I teach is to utilize precision in our explanations and definitions of topics in mathematics. The definition of a circle is the collection of points equidistant from another point. Students will be drawing their own circle in their math toolbox according to the definition of a circle using an index card scaled with the same units that appear in their gridded notebooks. By forcing students to think about the definition of a circle while creating a circle, the definition is more likely to "stick."

Students will then create a circle of equal size outside their toolbox to cut up and manipulate. Our discussion will include how to cut up the circle: wedges? Do we cut out a square and try to find the areas of the remaining pieces? If we were to cut up the circle into specific shapes, how would rearranging those shapes enlighten us as to what the area is? Since students have already seen the n-gons, one can funnel them into thinking about finding the area of those shapes. Many students will go to cutting it up like a pizza into many triangles. A circle can also be cut up into these wedges, although the wedges are not triangles.

Because we have established by now the concept of π, we can finally discuss area in depth. Here comes the exploration in how πr^2 emerges from the arrangement of the wedges of our circle. Color is utilized here to show where the circumference (shown below in red) appears in the arrangement, and where the radii (shown below in yellow) are. This makes it clear that the base of the "rectangle" is really just half the circumference since students can see half the color on top and half the color on the bottom.



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emphasis added by the boldface signals the emphasis that NCTM intends to give to the empowerment of each and every student through mathematics.

For, as mathematics teachers, we are engaged in something much bigger than the daily tasks of curriculum selection, instruction, and assessment. Mathematics is an essential analytical tool that we give to students to help them to better understand their context, experiences, and the world—and potentially to make the world a better place. Never has this been more important for each and every student and for our society. Many of our societal problems are increasingly formulated in mathematical terms, and their solutions frequently depend on mathematical understanding. Mathematics is essential not just to college and career readiness but also to informed and active members of our democratic society.

Without a strong understanding of mathematics and a positive mathematical identity and sense of agency, students are unlikely to have the tools necessary to make effective choices in their own lives. Furthermore, without deep mathematical understanding and positive identity and agency, students are unlikely to be able to understand and challenge many of the decisions and actions of those in power. The future of our democratic society depends on our ensuring that each and every student is empowered by the opportunities that mathematics affords. I encourage you to make a commitment this year to engage with your colleagues to make issues of equity, access, and empowerment part of your professional discussions, your conscience, and your daily actions. NCTM is committed to collaborating with others to make this vision a reality.

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Making Magic: Lessons from Improv

Melanie Valentine Durfee, Cedar Middle School

I am hopeful that every math teacher has experienced the joy of a perfect moment in the mathematics classroom. It could be a just a second, an episode, or an entire class period in which the students and teachers are in perfect synchronization, when all members of the class are building on each other's ideas. However, for most of us, this kind of magic in the classroom does not happen nearly often enough. Mathematics theorists Martin, Towers, and Pirie (2006) compare the management of this type of classroom interaction to improvisation techniques of music and theatre. They give three guidelines: 1) Don't write the script in your head, 2) Wait for a collective structure to emerge, and 3) Pay attention to the group mind. I will discuss each of these tenets and give examples of what it might look like in a mathematics classroom, drawing from my experience as a 7th grade math teacher.

1. Don't write the script in your head. I begin this paragraph with a caveat: The teacher does need to be aware of possible student misconceptions with several examples and non-examples at the ready. However, the teacher needs be in the mindset to be able to adapt the lesson seemingly effortlessly.

What it might look like*:* I was guiding my students to show adding and subtracting positive and negative integers using a life-sized number line. When two students were walking equal expressions [3 – (2) and 3 + 2], Caleb spoke out.

Caleb: They are doing it wrong. They are both walking the same numbers.

Me: Caleb, what do you mean?

Caleb: They can't both be doing the same thing.

James: Yes, they can. They mean the same thing.

Me: James, what do you mean by "they mean the same thing?"

James: Three minus negative two is the same as 3 plus two.

Caleb: Oh. It looks like they are dancing.

[waiting a beat]

Brandon: *Begins singing the instrumentals for the beginning of "The Lion Sleeps Tonight."*

Other class members join in the song.

my classes, the largest circle provided the least precise approximation of π!). Of course, at very large sizes of n-gons the M&M's could beging to give a good approximation, but the limitations of the sizes of circles students are working with do not allow for this precision in estimation. I then ask students: how can we find a closer approximation of π if simply measuring larger circles is ineffective in accomplishing this goal? The solution is not larger circles. The solution is smaller M&M's.

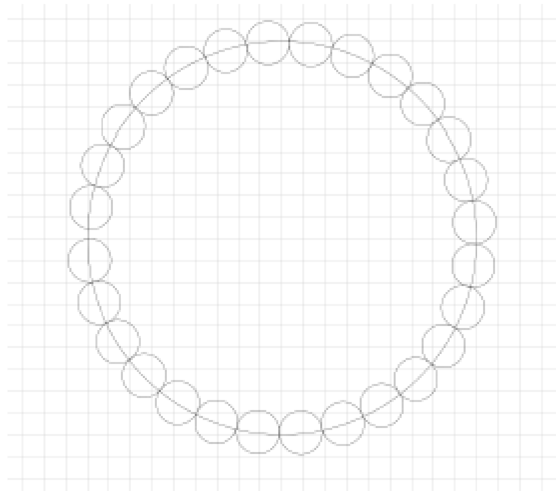
I have a Geogebra file ready to show n-gons with large numbers of sides and the decimal approximations that follow. These smaller and smaller n-gons represent smaller and smaller "M&M's." So, for example, I'll refer to the regular octagon in the file as "a circle that can be measured with 8 M&M's." The sizes of the M&M's used to measure the n-gon will vary, but the big idea is being able to measure with any size of unit. This is also a great way to introduce technology because manufactured food will never be a perfect circle, and we will never be able to line up the M&M's themselves perfectly around our circles. The only way to achieve the precision we need is to use technology. This piece is also where I want to educate students on some of the history of π and how humans first approximated it.

With our approximations, we can find more and more decimal places of π, but the approximations never seem to "end" or loop like our fraction decimals have. I will pose the question to my students: How many sides will our n-gon need in order to approximate the first 8 digits of π (around 10,000 sides!)? The first 10 digits (around 100,000 sides!!)? At this point the human brain isn't going to be able to tell the difference between a circle and a 100,000-gon, but the approximation for π still seems too "off" somehow. We can always keep getting closer and closer to π using regular polygons, but we never quite reach it. I slowly give students the table of values in terms of our M&M's:

Number of M&M's around the circumference	Number of M&M's across the diameter	Ratio of circumference to diameter
4	$\sim 1\frac{1}{2}$	~2.7 ...
8	$\sim 2\frac{3}{5}$	~3.08 ...
16	$\sim 5\frac{1}{8}$	~3.121 ...
32	$\sim 10\frac{1}{5}$	~3.137 ...
64	$\sim 20\frac{2}{5}$	~3.1403 ...
128	$\sim 40\frac{3}{4}$	~3.14127 ...
256	$\sim 81\frac{1}{2}$	~3.141104 ...
10,000	$\sim 3{,}183\frac{1}{10}$	~3.141592602 ...
100,000	~31831	~3.141592653073 ...

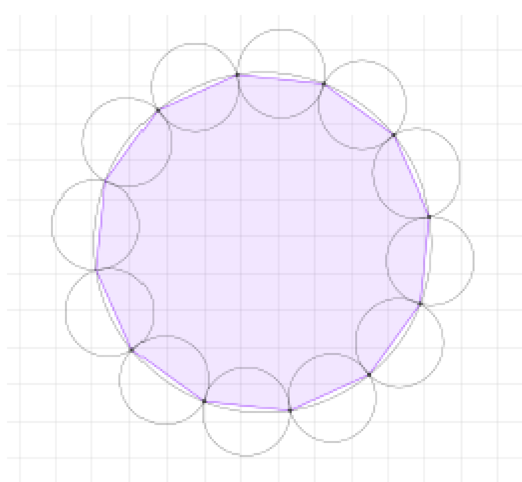
Table 1: M&M's of various sizes

It is important to stress here that, for example, exactly one and one half M&M does not fit across the circumference in the square. We are not sure exactly how much of an M&M the remaining space needs, so all we can do is estimate. Even though approximately 31,831 M&M's fit across the 100,000-gon, it's not exact. In reality, an irrational amount (31830.988236...) are needed. No matter how small our unit is, it can never perfectly measure the diameter of the n-gon. Students can see through this exploration that something is being approached, and the pattern that this number has is foreign. Can you guess the digits that a billion-gon will produce?



The circles are created to make these gaps noticeable to students, although due to the imperfections in the M&M's, some students may still find whole numbers of units to measure both the circumference and diameter of a single circle. This should not happen with the majority of students, however. This is a surfacing of incommensurability. This unit cannot measure both the circumference and diameter perfectly, but is there a unit that can? We know this does not exist for n-gons other than hexagons, but this is a deep question to be asking 7th grade students. It is a question they will no doubt encounter again in the coming years, especially when $\sqrt{2}$ appears later in the eight grade core. Understanding commensurability is a fundamental piece of understanding irrational numbers, and the overarching goal here is to build a foundation upon which students can start thinking about irrational numbers.

The discussion about the choice of M&M's to measure also leads to the use of regular n-gons in estimating π. When we "draw in" the diameters of the M&M's, however, it's clear that students have actually constructed their own n-gon:



Something fascinating emerges from this exploration. The worksheet has circles of various sizes, and yet the larger circles are not giving a better approximation of π (in the case of both of

Me *(after 15 seconds of song)*: Let's have two other students walk a different set of equations.

For several days after that incident, whenever our class discussion steered toward congruent expressions, the same song emerged. I have since led several other classes in that same activity that used a life-sized number line to show equal expressions. Sometimes students break into song; sometimes they do not. The song that emerges is never the same for any two classes.

2. Wait for a collective structure to emerge. If students are working on mathematical problems, which are both sufficiently challenging and within their reach, a structure will emerge (Martin et al., 2006).

What it might look like*:* After assigning my students six rather challenging problems to work on independently, I had to leave the classroom. When I returned ten minutes later, I saw that several students were out of their seats, and I heard a great deal of loud chatter. I observed unlikely groups of students working together on the problems. I was surprised by which students took on a leadership role and which students seem to prefer working on their own. This was a dynamic I did not expect to evolve. The students continued to work productively in this manner for another twenty minutes before their chatter evolved into mathematically unproductive comments.

3. Pay attention to etiquette of the group mind. Both students and teacher need to display two attributes to foster the group mind (Martin et al., 2006). The first is that everyone needs to be willing to pay attention to each other. The second is that all must be willing to alter their conceptions after listening to each other. Students intuitively may not know how to listen to each other; they must be taught how to do that.

What it might look like:

Me: Look at the number on your desk. If your number is an odd number, raise your hand. Odd numbered students explain to your table partner why adding a negative number to its zero pair makes zero.

Kaydee: Mrs. Durfee, I don't get what I am supposed to say.

Me: What happens when you add positive 3 and negative 3?

Kaydee: Oh, I get it. It makes zero because their cancel each other out.

Me: Right Kaydee. Now explain that to your table partner.

Me: Look at the number on your desk. If your number is an even number, raise your hand. Even numbered students, listen to your table partner's explanation again and be prepared to share it with the rest of the class.

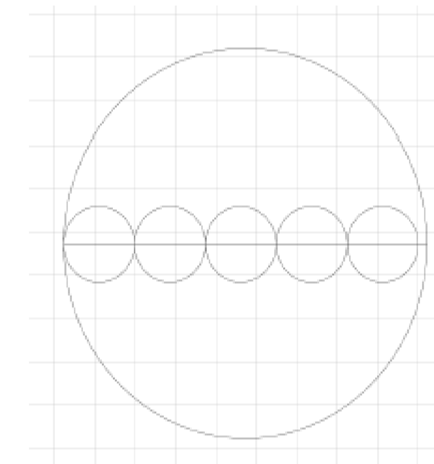
A teacher across the hall from me uses a different method to help students talk and listen to each other. On her wall is a poster with the following open-ended statements:

> I agree with what _______ said because ________.
>
> I want to add on to what ________ said. I think that ________.
>
> __________, can you explain your thinking?
>
> I disagree with _______ because __________.
>
> I have a connection to what ______ said. It is ______________.

When this teacher conducts class discussions, she requires students to use one of these five sentences to frame their comments. Following this format gives a structure for students to listen to each other and make productive mathematical comments.

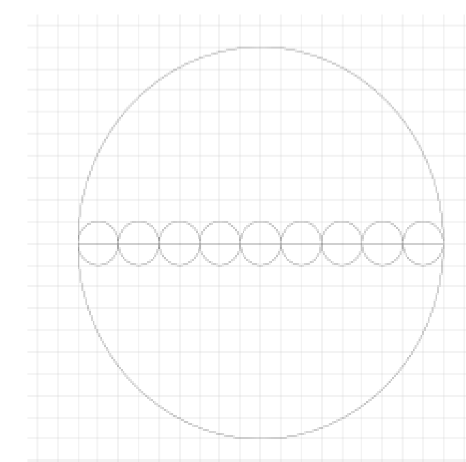
Using improv techniques to create classroom unity is motivating to students, because it gives them what they both need and want: a sense of belonging to a group. The group to which they belong, their math class, can accomplish much greater understanding together. When I am able to teach in this manner, I find that I do not need outside motivators to keep students' working. I do not have to rely on prizes or rewards to keep students attentive. Being part of the group seems to be reward enough. Teaching students who want to learn is nothing short of magic.

Martin, Lyndon, Jo Towers, and Susan Pirie. "Collective mathematical understanding as improvisation." *Mathematical Thinking and Learning* 8, no. 2 (2006): 149-183.



...there is an ever so slight gap. This unit cannot measure our circle perfectly. I ask students to estimate how much this gap might be. In this case the measurement would result in approximately five and a fifth of our unit.

On a different circle, our unit measures the diameter perfectly:



...but now the unit cannot perfectly measure the circumference:

A Lesson on IncoM&Mensurability

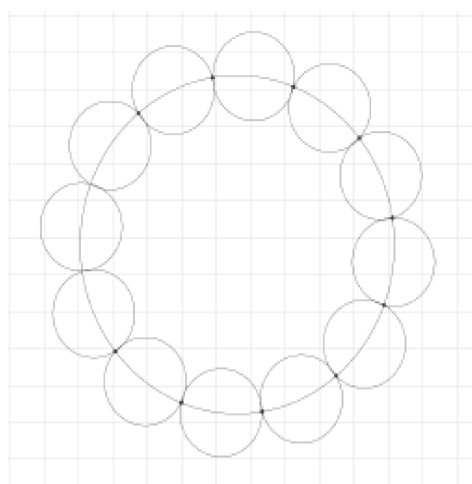
1. Exploration of terminating and repeating decimals

   Before the lesson on circles, a quick re-examining of decimal expansions must take place. It is important that this piece comes before the need for (and thus introduction of) π. Students will be given as homework the task of "finding a fraction whose decimal expansion never repeats or terminates." Students are allowed to use calculators.

   Even though there are expansions that are too long to repeat on the calculator screen, students must be able to interpret that the decimal expansions are forever repeating. We will explore this by both writing some decimals out by hand using long division and also using technology for the fractions with larger denominators. I am seeking to engage students by tackling the misconception head on, since student motivation seems to spike when the goal is to disprove a teacher.

2. Using M&Ms to measure circles.

   Students are to use M&M's and their diameters to estimate the circumference and diameters of various sizes of circles. The reason we are using M&M's is because it's "easier" to measure circles with the diameters of smaller circles. Attempting to arrange squares around the larger circle becomes cumbersome because of the overlapping edges. Students' goals in arranging the M&M's along the circle is to have the circumference and diameters of the circle aligned as best they can with the diameters of the M&M's, like so:



   The diameter of the M&M's are our "unit" for today. Therefore the measurement of the circle above with the M&M's is a total of 12 "units." Our unit measures the circumference of the circle perfectly. When we go to measure the diameter, however:

Improving Elementary Students' and Pre-service Elementary Teachers' Attitudes and Knowledge Related to STEM Subjects through an Enrichment Robotics Program

Elaine Tuft & Vessela Ilieva, Utah Valley University
Jay Jayaseelan & Jaylene Ahlmann, Learning Through Robotics, LLC

Abstract

A partnership between a school of education, a school district, and an educational robotics company are formed to provide enrichment robotics classes for elementary students with the intent of improving attitudes toward and knowledge of STEM subjects of both elementary students and preservice elementary school teachers.

Keywords: STEM Education, Robotics, Enrichment Classes, Elementary Education

Introduction

Nationally, there has been an emphasis on science, technology, engineering, and mathematics (STEM) education with the purpose of better preparing K-12 students to enter and succeed in higher education as well as to prepare them for careers that pay well and for which there will be a continual and growing need. However, many elementary and secondary students do not like mathematics and do not feel confident in their abilities to be successful in this subject, let alone with its applications in science, technology, and engineering (Ma & Kishor, 1997). This dislike, disinterest, and lack of confidence often begin in elementary school. When children leave elementary school, most of them have already decided if they are good or bad at mathematics and science and whether they are going to pursue those subjects much or not. Further, many preservice elementary school teachers also do not feel confident in their knowledge of STEM subjects and their ability to teach these subjects well (Jong & Hodges, 2013). They are nervous about the prospect of teaching these subjects, and some say that they hope they will teach a lower grade where they think their lack of knowledge in these subjects won't be as consequential or apparent.

To address these concerns, a partnership was formed to offer enrichment robotics classes for 5th and 6th graders taught by preservice elementary school teachers. Elementary school was chosen because of the need to pique the students' interest in STEM subjects early. Robotics classes were an attractive option to address this problem because they are very hands-on and engaging. They also integrate all four of the STEM subjects. Additionally, seeing and successfully experiencing applications of mathematics help students feel more confident in their own mathematical abilities. Those experiences also help them see how mathematics is connected to other subjects, and this leads to more positive attitudes toward mathematics.

One purpose of this project was to increase elementary- and college-age students' knowledge of STEM subjects and applications through the use of Lego Mindstorms™ robotics. A second purpose was to provide elementary education majors more STEM-based experience teaching children prior to receiving their teaching license. The final purpose was to improve the participating college and elementary students' attitudes related to STEM subjects, particularly in relation to mathematics applications for engineering, programming, and building skills used in robotics and technology.

Review of the Literature

The importance of the preparation of our students in mathematics and other STEM subjects has long been advocated (NCTM, 2000, 2014; National Research Council, 2011). Many studies have pointed to the need for students to be more prepared for STEM subjects and subsequently for more STEM careers (DeJarnette, 2012; PCAST, 2010). According to the National Math and Science Initiative (NMSI)—launched in 2007 by leaders in business, education, and science to reverse the decline in U.S. students' math and science educational achievement—studies show that 69% of high school graduates are not prepared for college-level science. Likewise, 57% of high school graduates are not prepared for college-level mathematics (NMSI, 2012). This limits their options of majors and subsequent careers or requires additional time in their college preparation for careers. This discourages many from even considering STEM careers, and as reported by NMSI, of the 30 fastest growing occupations through 2016, sixteen will require substantial mathematics or science preparation (NMSI, 2012).

"randomness multiplied by randomness will always result in randomness." Yet, I'm sure that if the question were altered by removing the word "different," most of the participants would have correctly answered "yes," thinking of $\sqrt{2} \cdot \sqrt{2}$. Why is this "randomness multiplied by randomness" suddenly resultant in something rational? It's important to think of an irrational number not just by its decimal expansion, but also by what it represents as a number.

In introducing any topic in mathematics it's important to build upon what is already known. 7th graders are familiar with decimals, so that is the form I will be focusing on. It is my goal to approach irrational numbers in a way that paves the way to understanding irrational not just as a decimal expansion but also as a concept through examining the incommensurability of two lengths (circumference and diameter of a circle) and definition (the collection of points equidistant from a given point). This approach is called "pre-exposure," a technique by Eric Jensen presented in the book Teaching With the Brain in Mind.

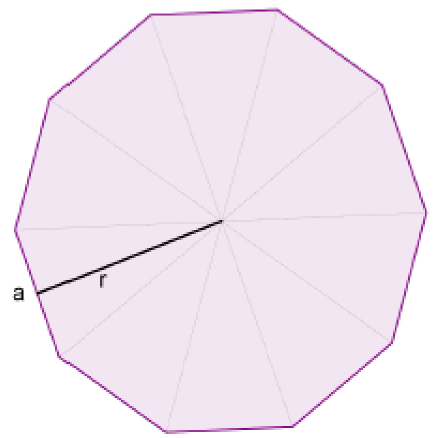
Our students require repetition in order to learn. Repetition can come from hearing, seeing, or practicing the same or similar topic over and over again. In order for repetition to not lose its "edge" in learning, teachers must plan strategically for upcoming topics. For some topics, this means a quick activity a few minutes before the lesson. For others, pre-exposure is more fitting. Pre-exposure occurs not minutes or hours, but weeks, months, or years ahead of time. By pre-exposing students to irrational numbers, I am setting them up for an entire year of potential thought. This "taste" of irrational numbers should begin to build a bridge between what they previously understood about numbers to an entirely new set of numbers that literally filled in the holes of previous mathematical thinking.

Jensen also heavily suggests similar information being presented in several ways. Research about using Geogebra as a tool is incredibly promising. Mental models (graphs, geometric representations, diagrams, representative models, etc.) are vital for many topics in math. In a theoretical framework by Lingguo Bu, Michael Spector and Erhan Slcuk Haciomeroglu, it was found that Geogebra aligns itself well with model-facilitated learning goals. The big idea in this framework is that "models are further utilized as entry points to support sense-making within a world of increasingly abstract mathematical ideas." Geogebra is a viable tool for these models as it provides many opportunities for all kinds of dynamic modeling.

Johnson-Laird created a theory of comprehension using models, regarding how learners have this first impression of ideas that immediately manifest themselves into a mental model from which to refer. These models can be manipulated to support deeper thought. For example, having a mental model for transformations of the plane can extend itself to transforming all kinds of functions. This means, however, that not only must the model be able to be dynamically manipulated in such a way to begin with, but it must be built upon a mathematically sound foundation. Not all mental models have the capability to extend further, which is why it's important to help guide these mental models ourselves as teachers. Succinctly said: "our understanding of a mathematical topic is a matter of having a functional mental model for it."

This is why I am integrating Geogebra into several aspects of my lesson. Circles are, of course, heavily rooted in geometry. It would be a huge missed opportunity to not include models such as regular polygons, graphs, and dynamic representations of manipulatives we do in class. I will be utilizing the CRA (concrete-representational-abstract) sequence, modeling with Geogebra physical models that students have already explored.

n-gon can be thought of as the sum of all the areas of the wedges. As n approaches infinity, this sum will converge to the area of a circle. There are n wedges in an n-gon, and each wedge has area $\frac{1}{2}ar$:



r is the length of the altitude from the center of the n-gon to the outer edge. Multiplying this quantity by n, $\frac{1}{2}anr$ gives the area of the entire n-gon. Because an is the perimeter, now we can finally use the constant of proportionality, π, to describe the "perimeter" of a circle (the circumference), and note that the perimeter of the n-gon approaches $2\pi r$ as n approaches infinity. Thus, $\frac{1}{2} \cdot 2\pi r \cdot r = \pi r^2$.

What Does the Research Say?

In beginning the journey of teaching topics involving irrational numbers, it's important to also self-reflect on our own understandings of irrational numbers. Natasa Sirotic and Rina Zazkis examined the relationship between 46 teachers' intuitive vs formal knowledge of irrational numbers in a 2007 study. Although the responses of teachers were evaluated based on "intuitive notions" rather than information recall from the teachers' time in formal math classes, there was a deficit in the group's ability to rigorously discuss irrational numbers overall. It is also important to note that the amount of undergraduate math courses taken and the time at which the teachers had completed them did not significantly affect their responses.

About half of the teachers had difficulty in discussing the probability of selecting a rational number in the pool of real numbers in the interval from $[0, 1]$. An alarming number of teachers (around 64%) either had no answer or claimed the probability was .5. Around a quarter of all teachers either chose 0 (the correct answer) or close to 0 as their answer, and nearly half of the teachers correctly identified the irrational numbers as "richer" than the rationals.

It is clear from the data that misconceptions are high, and students' own misconceptions possibly stem from teachers' own misunderstandings about irrational numbers. In reading the responses of many of the teachers, it's clear that although there is a familiarity with this set of numbers, the depth of understanding remains rather shallow. A fifth of the participants could not name an irrational number besides π, e and $\sqrt{2}$.

There is an inability to divorce an irrational number from its decimal approximation that persists in many of the responses in the study, in particular the responses to the question "Is the product of two different irrational numbers always irrational?" About half said yes, many with an explanation of

This improvement in students' preparation in STEM subjects needs to begin in elementary school. Afterschool and enrichment programs have shown promise in helping students become more excited and confident about STEM subjects (Afterschool Alliance, 2013a; Afterschool Alliance, 2013b; Krishnamurthi et al., 2014; Mohr-Schroeder et al., 2014; National Research Council, 2011).

The NMSI has also stressed that teachers need more training, especially in STEM subjects. This is consistent with the recommendations of other organization recommendations (NCTM, 2000, 2014; NMSI, 2012; PCAST, 2010). Additional training and experience will enable them to be more confident teaching these subjects.

Project Description

This project is a partnership between a large university's School of Education, a local, sizeable school district, and an educational robotics company to provide enrichment robotics courses for 5th and 6th graders. Robotics classes are taught quite often in secondary schools, but they are less common in elementary schools. However, probably the most distinct aspect of this project is that preservice elementary teachers serve as instructors for the robotics classes. Personnel from the robotics company train students majoring in elementary education how to use and teach classes with customized robotics equipment including Lego Mindstorms™ robotics. The training occurs on Fridays in the School of Education building. The university students then go to the participating elementary schools to teach enrichment robotics and programming classes to 5th and 6th graders Mondays through Thursdays. The program consists of a ten-week course in which classes are taught once a week. Most of the classes are held after school, but a few are taught during the regular school day. For the first iteration of the program, it culminated in a robotics showcase sponsored by the school district. So far, there have been five offerings of this program.

Methods for Assessing Impact of Program

One way to measure the impact of the program is by looking at the numbers of elementary schools, elementary students, and university students who participate in the program. The number of courses taught also gives some indication of its impact. We were also interested in how the program might affect participants' attitudes and knowledge. The participating elementary students were given a knowledge-based test related to STEM subjects, focusing on application mathematics and Lego Mindstorms™ robotics. The assessment was administered both before and after participating in the classes. Some of the 5th and 6th grade mathematics areas were covered such as figuring the distance a robot would travel, area, angles, decimals, and fractions. It also contained questions related to the programming of the robot to make it travel certain distances and directions. There were 20 separate items recorded as correct or incorrect for the test. There were 330 students for whom we were able to obtain both the pre- and post-knowledge-based test as well as the parental consent forms and the student assent forms. The percentage of students who answered an item on the test for this group of students was compared from the pre-test to the post-test with paired *t*-tests to see if there was a statistically significant difference.

The elementary students were also given a survey designed to learn about their attitudes related to STEM subjects and careers as well as educational robotics both before and after participating in the classes. For this instrument, they had to rate certain statements such as, “I am good at math” from 1 (meaning very unlike me) to 5 (meaning very much like me). We had a complete set for 279 students with this instrument to use in this part of the analysis. The average responses pre vs. post were compared for each item using *t*-tests to see if there was a statistically significant difference.

The impact of this program was also assessed through anecdotal data and informal interviews with the enrolled elementary students, the preservice elementary teachers who served as the instructors, principals, parents, and district personnel.

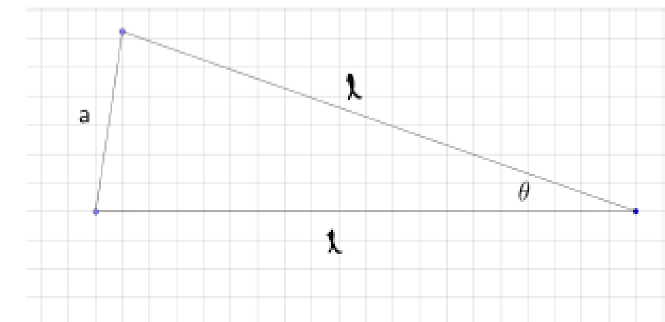
Results

Elementary Students

The elementary students were very positive about their experiences in the robotics classes. They were engaged and active during the classes. One elementary student described his experience this way: “It was really interesting to make different types of robots do cool things with sensors. Programming robots was challenging and made me think. I've never done anything with robots before, but I really liked it, and I would like to do more.”

As program administrators, one of the most rewarding results of the program was the effect it had on the students’ attitudes and understanding of problem solving and persevering in finding solutions. One participating female elementary student said about constructing, programing, and testing her robot, “It probably won’t work the first time, but that’s OK. It’s not failing; it’s learning what to do, how to change it to make it work the next time.”

exploration above. Recall the image of one wedge of an n-gon with radius l and each edge measuring a:



The expression for the ratio of the circumference to diameter of an n-gon was evaluated to be $n \cdot sin(\frac{\pi}{n})$. To find the length of a, this ratio can first be multiplied the diameter ($2l$) to find the entire circumference, and then divided by n (the total number of sides) to get each side length a. This results in the expression $2lsin(\frac{\pi}{n})$.

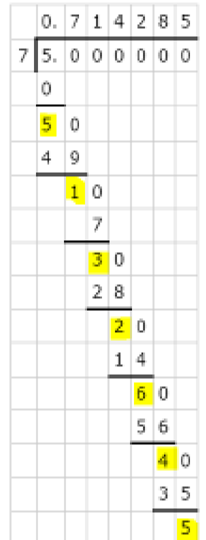
When does this expression for a result in a rational number? Let's first examine $sin(\frac{\pi}{n})$. Since n is a natural number greater than 3, $sin(\frac{\pi}{n})$ will be irrational for all values of n other than $n = 6$ (this is known to be true by Niven's theorem, which states that the only values for θ, $0 \le \theta \le \frac{\pi}{2}$ such that $sin\theta$ is rational are $\theta = 0, \frac{\pi}{6}, \frac{\pi}{2}$). $n = 6$ is a special case of n-gon, since a regular hexagon is composed of 6 equilateral triangles. It is the only n-gon for which the side length a is commensurable with the diameter l. Beyond that, however, something interesting happens. Take $n = 10$ for example. $sin(\frac{\pi}{10})$ evaluates to the irrational number $\frac{1}{4}(\sqrt{5}-1)$. If l is any rational number, then $2lsin(\frac{\pi}{n})$ will still result in an irrational number. The two lengths a and l are therefore incommensurable. If a ends up being a rational number itself, then its expression $2l \cdot \frac{1}{4}(\sqrt{5}-1)$ must evaluate to be rational. The only way for this to be rational is if l were the conjugate of $\sqrt{5}-1$, an irrational number. And in fact, evaluating the expression for $l = \sqrt{5}+1$ gives:

$$2(\sqrt{5}+1) \cdot \frac{1}{4}(\sqrt{5}-1) = \frac{1}{2}(\sqrt{5}+1)(\sqrt{5}-1) = \frac{1}{2}(5-1) = 2$$

...a rational number. l could also be equal to $\frac{1}{k\sqrt{5}-1}$, $k \in \mathbb{N}$ to produce a rational length a, however this value will also always be irrational as well. There is no rational number for l that can produce a rational length a.

For $n > 6$, both of the lengths cannot coexist as rational numbers. Because of this, the lengths a or l can continue to be "cut up" into smaller and smaller pieces in an effort to find a unit that can measure them both, and yet that unit can never be attained. The circumference and diameter are incommensurate. This is the link between π in the 7th grade, and $\sqrt{2}$ in the eighth grade. $\sqrt{2}$ was widely "known" to be irrational long before this was also known about π. Euclid's ideas of incommensurability were more easily utilized on the diagonal of a square with side lengths of 1 than on the circumference and diameter of a circle. Using the sides of n-gons as measurements, however, reveals this incommensurability to students who may not have experienced incommensurability before, and prepares them for thinking about the topics in the eighth grade.

All these pieces of prior knowledge and exploration are slowly revealing to our students how to begin to understand this constant. Without the constant, properties of circles can be observed but not concretely measured. With it, now we can finally approach the area of a circle. The area of an



The algorithm actually cycles through all the possible remainders of 7, $\{1, 2, 3, 4, 5, 6\}$, before coming back to a remainder of 5. Now the cycle will start over, and the infinite, predictable nature of $\frac{5}{7}$ can be observed. Because an irrational number's decimal expansion does not terminate nor repeat, we can conclude that it cannot be written as a ratio of two integers. As teachers we must distinguish for our students the understanding of rational numbers from their introduction of irrational numbers. The properties of circles introduce a brand new form of infinite decimal expansion in the form of irrational numbers. Using n-gons to approximate π is surfacing new ideas of what a decimal expansion can be by providing no "pattern" or algorithm with which to predict the next iteration. The only way to find the next closest approximation is to continue to create n-gons with more and more sides.

Knowing that a number can be written as a ratio of two integers opens the door for a property middle school students are familiar with: a common denominator. Being able to write two rational numbers with a common denominator allows students to geometrically compare the size of the two numbers relatively easily, "cutting up" each rational number in such away that they are composed of the same-sized pieces. Early understanding of irrational numbers began geometrically, with $\sqrt{2}$ being the length of the diagonal of a square with side lengths of 1. This magnitude was different from other magnitudes; it could not be easily compared to other whole or rational numbers.

Euclid approached irrationality in his Elements through incommensurability, that is, "Those magnitudes are said to be commensurable which are measured by the same measure, and those incommensurable which cannot have any common measure." The big idea behind commensurability is the idea that two numbers are commensurable when they have a common unit with which to measure them both a whole number of times. $\frac{3}{8}$ and $\frac{1}{7}$, for example, are commensurable since $\frac{1}{56}$ is a "unit" that can measure each of these numbers with a whole number. $\frac{3}{8}$ can be written as $\frac{21}{56}$, and is thus measured by $\frac{1}{56}$ 21 times. $\frac{1}{7}$ can be written as $\frac{8}{56}$, so it is measured by $\frac{1}{56}$ eight times. We teach our students that for any two rational numbers, a common denominator can be found. The implication of this is that any two rational numbers are commensurable. A big idea behind irrational numbers is that there is no common measure that can be used to measure both an irrational number and a rational number. $\frac{1}{3}$ and $\sqrt{2}$ cannot both be "cut up" in such a way that a number can measure them both a whole number of times. $\sqrt{2}$ being incommensurable with rational numbers is a huge mathematical discovery that happens in the eighth grade, and this reasoning can be beautifully surfaced through the n-gon

The elementary students' knowledge related to robotics and the mathematics and technology used in programming the robots improved significantly. For the knowledge-based test, there was a statistically significant improvement in the percentage of students who answered the item correctly for 19 out of 20 problems. For the other problem, a high percentage of the students answered it correctly on the pretest; therefore, there was not room for significant increase.

In the first administration of the attitudinal survey related the STEM subjects and careers, many of the students were quite positive in their attitudes, which was not too surprising since at most schools, students opted to enroll in this program as an after-school class. Therefore, there was also not room for statistically significant improvement for most of the items. However, a t-test for the difference between dependent means was conducted (paired sample) for each item, and there were two items in the survey for which there was significant improvement. The first was, *I know a lot about robotics*, $t(393) = -11.6583$, $p < .001$, $R^2 = .257$. The second was, *I'm good at programming Lego Mindstorms™ Robotics*, $t(393) = -15.6631$, $p < .001$, $R^2 = .384$.

University Students

The preservice elementary teachers who served as instructors of these robotics classes also felt like it was a valuable experience. One instructor said, "I really enjoy watching the students get excited to be a part of this program. I love watching them come into the classroom ready to participate and learn all that they can." Her comment was illustrative of other sentiments expressed by the participating university students. They also recognized the value of the program, as expressed by one of the other university students, "Teaching students to love learning is vital to the future of our society. We need programs like the robotics class that transfer knowledge into doing. If we can inspire these youth and push their understanding, our society will benefit — they are the future."

One of the most gratifying effects of the program on the preservice elementary teachers was the increased confidence it gave them in teaching STEM subjects. One of these students said, "When I first began, I was intimidated by all of the parts and programming that were involved. I did not have a lot of knowledge in working with STEM subjects, and this experience has allowed me to become skilled at teaching more difficult subjects as well as gain more self-assurance in teaching subjects that many of us often shy away from." Another preservice elementary teacher described the effect of participating in this project on her this way, "This experience will allow me as a teacher post-graduation to bring the passion and love I myself have developed for STEM into the classroom. The confidence I have gained while teaching in the robotics program will help me incorporate STEM into my teaching in a way that will allow my students to learn important concepts in a hands-on and interactive manner."

Other Stakeholders

The principals of the participating schools all spoke highly about the program. They each expressed the desire to offer the courses again in their schools. The parents who were informally interviewed also had positive things to say about the program. For example, one parent said,

> It's a great program as far as teaching social skills and getting [my son] away from just looking at a screen to actually working physically with a computer. He's shown some interest in construction, but whatever he does, this program is great for making him a more well-rounded and confident individual.

Reach of Program

Thus far, these robotics classes have been held in 25 different elementary schools. There have been 79 courses in these schools that have been taught by the university students. Approximately 2,370 elementary students have been enrolled in these classes. There have been 26 university students (most of them elementary education majors) teach these classes, and 12 of them have taught the course more than one semester. Many other university students have expressed interest in teaching these classes more than one semester, but their school schedules and obligations have conflicted with the times the classes have been offered.

Importance for the Field

The impact of this program has great educational importance with its promise of helping elementary students become more excited about STEM subjects, perhaps earlier on than they would have. If students become more excited about STEM subjects in elementary school, they are more likely to be interested in STEM subjects in secondary school and continue pursuing opportunities to learn about them. They are also more likely to participate in other STEM-related activities. This will help them be better prepared to enter college ready to enroll in college-level STEM classes.

The importance of this program is also great in helping prospective elementary school

of three. By this point students concluded that the same picture would be drawn over and over again forever, and finding one third of the entire square meant adding one of each group of three: .3+.03+.003+.0003... in an infinite series.

Many of the decimals (both repeating and non-repeating) students have been exposed to up until this point can be recreated by decomposing a flat. Simply cutting up the wedges of an n-gon into smaller pieces does not properly show that π is irrational to students in middle school, since students have already experienced an infinite decimal expansion as a result of continuously "breaking up" a figure to become more precise. As teachers we must distinguish for our students this experience from their introduction of irrational numbers. The properties of circles introduce a brand new form of infinite decimal expansion. Using n-gons to approximate π is surfacing new ideas of what a decimal expansion can be by providing no "pattern" with which to predict the next iteration. The only way to find the next closest approximation is to continue to create n-gons with more and more sides. So we must address with our students why rational numbers *can* be predicted effectively.

I'll start with a claim: all rational numbers can be written as either a terminating or repeating decimal. We have already examined the conditions in which a rational number's decimal terminates. For all rational numbers whose decimal expansions repeat, we must examine the division algorithm. Using this algorithm, there are two iterations that could happen in each "round" of the algorithm: either the remainder is 0 and the decimal terminates, or the remainder is not 0. If we examine all the possible remainders of b in a rational number $\frac{a}{b}$, they are elements of the set $\{1, 2, ..., b-1\}$. Thus, in each iteration there are two occurrences if the remainder is not 0: either the remainder is an element of the set, and a new iteration of the division algorithm begins. Or, a remainder that has already happened occurs, and the division algorithm repeats a cycle. The most digits a cycle can have is $b-1$ since there are $b-1$ possible remainders.

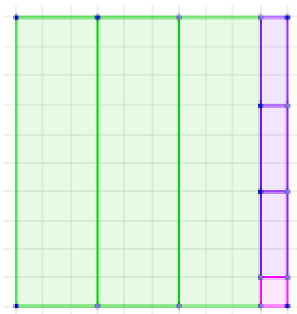
Take $\frac{5}{7}$ for example. The discussion of the division algorithm is important with students since when typing $\frac{5}{7}$ onto a calculator, the screen shows: .7142857143. To many students, the pattern here is lost due to the limited size of the screen and the rounding. A deeper understanding of rational numbers means helping students understand decimal expansions past what the calculator can show, which is where the division algorithm comes into play. Doing three iterations of the division algorithm:

	0.	7	1	4
7	5.	0	0	0
	0			
	5	0		
	4	9		
		1	0	
			7	
			3	0
			2	8
				2

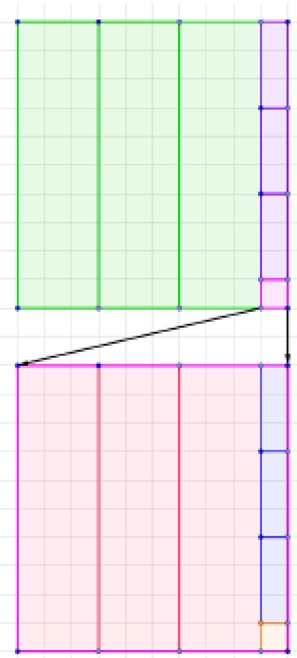
...only 3 remainders (highlighted) have been used so far: 5, 1 and 3. If the next remainder is 5, the cycle will repeat again. If the next remainder is 1, then the algorithm gets in a loop alternating between 1 and 3 forever. Here's what really happens:

10's, 100's, 1000's, etc.? It's clear to students visually why one hundred hundredths cannot be cut into 8 equally sized groups of hundredths, because 8 does not divide 100. If we look at one thousand thoudandths, however, now one whole can be cut into 8 equally sized pieces of one hundred and twenty five hundredths. This can also be interpreted as 12.5 hundredths. Either way, $\frac{3}{8}$ can be written as .375 as it is 3 copies of $\frac{1}{8}$. This is a student's view of $\frac{3}{8}$. A more rigorous way of thinking about this fraction is in terms of the process outlined in the previous paragraph. $\frac{3}{8} = \frac{3}{2^3}$, so to write the base as a power of 10, the fraction needs to be multiplied by $\frac{5^3}{5^3}$. $\frac{3}{2^3} \cdot \frac{5^3}{5^3} = \frac{3 \cdot 5^3}{2^3 5^3} = \frac{375}{(2 \cdot 5)^3} = \frac{375}{10^3}$.

To help students understand both terminating and repeating decimals, it is still vital to connect both varieties of decimals to the decomposition of a flat (a 10x10 square). Earlier in the year, we explored the decimal expansion of $\frac{1}{3}$ by taking a ten by ten square and attempting to cut it into equally sized pieces like so:



There is an extra hundredth (shown in pink) that was not able to be included in either group of three (the group of green three tenths or the group of purple three hundredths). So, we "zoom in" on the pink hundredth and start the process over again:



Now the red pieces are each three thousandths, and the blue pieces are each three ten thousandths. A similar situation occurs, where a single ten thousandth was not able to be included in either group

teachers become more knowledgeable in aspects of STEM subjects and more confident in teaching them. If more elementary teachers are competent teaching STEM subjects and enjoy teaching them, their elementary students will likely enjoy those subjects more. As the program continues, assessment will provide more information about how to refine and improve the program as well as how to increase its reach—with elementary students, future elementary teachers, and the community.

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a is the length of each edge of the n-gon ($a = \sqrt{2}$ in the square pictured above, for example). Using the law of cosines to calculate a, the length of a is $\sqrt{2-2cos\theta}$. In a regular n-gon, the size of theta will be $\frac{2\pi}{n}$, and the circumference is the expression for a multiplied by n. Putting all the pieces together to find the ratio of circumference to diameter (the diameter in this case being 2), the "π approximation sequence" is $a_n = \frac{n}{2}\sqrt{2-2cos(\frac{2\pi}{n})}, n > 3$ (it doesn't make sense to talk about a regular polygon with less than 3 sides).

It takes a large value for n to approximate the first 8 digits of π. Around 10,000 sides, the approximation reaches up to 3.1415926 in π. As a larger value for n is used, the approximation becomes more precise, and yet the decimal expansion never seems to completely reach a definite number.

To truly claim that this sequence does indeed converge to π, however, the limit of the sequence as n approaches infinity can be evaluated. The sequence can be rewritten using the double angle formula for the cosine function as such:

$$a_n = \frac{n}{2}\sqrt{2-2cos(\frac{2\pi}{n})} \implies a_n = \frac{n}{2}\sqrt{2-2(1-2sin^2(\frac{\pi}{n}))} \implies a_n = \frac{n}{2}\sqrt{4sin^2(\frac{\pi}{n})}$$

$$\implies a_n = n\sqrt{\frac{4sin^2(\frac{\pi}{n})}{4}} \implies a_n = n\sqrt{sin^2(\frac{\pi}{n})} \implies a_n = n|sin(\frac{\pi}{n})|$$

Taking the limit of the sequence should now be a bit simpler. To find this limit, the sequence can be rewritten as a fraction so L'Hospital's rule can be utilized. To avoid taking the derivative of $|sin(\frac{\pi}{n})|$, it can be noted that for values of $n > 3$, $|sin(\frac{\pi}{n})|$ and $sin(\frac{\pi}{n})$ have identical range. Thus, $|sin(\frac{\pi}{n})|$ can be replaced with simply $sin(\frac{\pi}{n})$:

$$\lim_{n\to\infty} n\, sin(\frac{\pi}{n}) = \lim_{n\to\infty} \frac{sin(\frac{\pi}{n})}{\frac{1}{n}} = \lim_{n\to\infty} \frac{cos(\frac{\pi}{n})(-\frac{\pi}{n^2})}{-\frac{1}{n^2}} = \lim_{n\to\infty} cos(\frac{\pi}{n}) \cdot \pi$$

Because $\lim_{n\to\infty} cos(\frac{\pi}{n}) = cos(0) = 1$, the limit of the sequence is simply π.

We can conclude then, that the limit converges to this constant of proportionality. However, this reveals little about the nature of this constant, particularly its decimal expansion. At no point in using n-gons to approximate π can we stop and have experienced how to generate the remaining digits without utilizing a larger value for n. That is, there isn't a concrete pattern to follow. Before the introduction of π, earlier in the year 7th graders explored several decimal expansions (both terminating and non-terminating) whose patterns can be generated, and thus are predictable as the decimal expansion continues (for example, $\frac{5}{7}$ repeats the digits 714285 in its infinite decimal expansion). The first decimal expansions that students learn about (even before the 7th grade) are terminating decimals.

The decimal expansion of a fraction will terminate if that fraction has a denominator that can be factored into $5^m 2^p, \ m, p \in \mathbb{W}$. This is because our number system is base ten, so the denominator of each decimal place value will be a power of 10: $10^n, \ n \in \mathbb{N}$. Because 10 has factors 5 and 2, this can be rewritten as $(5 \cdot 2)^n \leftrightarrow 5^n 2^n, \ n \in \mathbb{N}$. A fraction of the form $\frac{a}{5^m 2^p}, \ a, m, p \in \mathbb{N}, \ \ m > p$ can be written in base ten: $\frac{a}{5^m 2^p} \cdot \frac{2^{m-p}}{2^{m-p}} \leftrightarrow \frac{2^{m-p} \cdot a}{10^m}$. The same process is done for $p > m$, multiplying the fraction by $\frac{5^{p-m}}{5^{p-m}}$ instead.

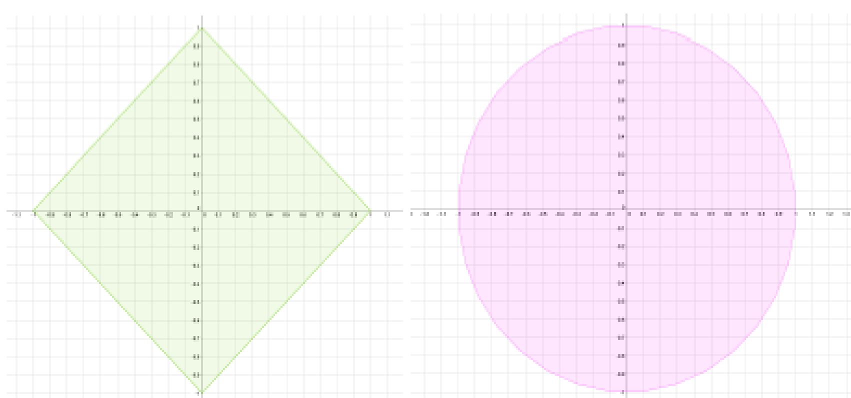
A fraction like $\frac{1}{3}$ can never be written with a base ten denominator as there is no way to write a power of 10 with a factor of 3 using a natural number in the numerator (since of course one could write $\frac{1}{3}$ as $\frac{\frac{10}{30}}{10}$). Students understand this through questions such as: Can we write $\frac{3}{8}$ in terms of

$$\frac{A_1}{A_2} = \frac{d_1^2}{d_2^2} \leftrightarrow \frac{A_1}{A_2} = \frac{(2r_1)^2}{(2r_2)^2} \leftrightarrow \frac{A_1}{A_2} = \frac{4r_1^2}{4r_2^2} \leftrightarrow \frac{A_1}{A_2} = \frac{r_1^2}{r_2^2}$$

This lemma holds even without knowing what the constant of proportionality is. All this shows us is that there is some α such that $\frac{A_1}{A_2} = \frac{\alpha}{\alpha} \cdot \frac{r_1^2}{r_2^2}$. Therefore to fully understand the area of a circle, one must have an understanding of this constant and how to find it.

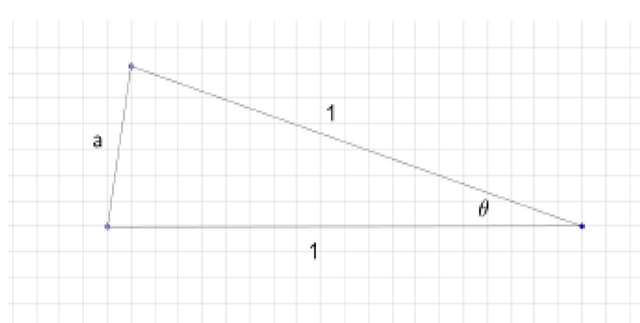
This constant of proportionality had been previously explored in history, but had not yet been refined into a universally agreed upon concrete number. The Babylonians approximated π by knowing there was "some ratio" involved in the circumference and area formulas, and used these formulas to solve for π. This method resulted in an approximation of simply 3. They were close but by no means precise. The Egyptians used a six-sided regular polygon to approximate this ratio to be 3.125, closer than the Babylonians but still imperfect.

To get a picture of how a middle school student might be exposed to this constant, consider the following example that surfaces for students the idea that this constant is infinite but in an entirely novel way. Archimedes built upon the Egyptians' idea of using regular n-gons to produce more and more accurate approximations of π. Start first with a familiar, but basic n-gon, a square centered at the origin with "radius" 1:



A square has a ratio of circumference to diameter of $2\sqrt{2}$, approximately 2.828. Pictured next to the square is a 32-gon also centered at the origin also with radius 1. Although the 32-gon looks rather circular already, the approximation this polygon gives is only about 3.13655, accurate to 1 decimal.

To test and see how many sides the regular polygon needs to more accurately approximate π we will think about each individual "wedge" of a regular polygon with radius 1:



A Review of Literature: Assessment Literacy for Mathematics Teachers and Students

Sheryl J. Rushton, Weber State University

Abstract

Over the past several decades there has been an emphasis in educational research on student assessment and achievement in mathematics. Formative assessments are designed to inform the instructional decision making process and require assessment literacy to interpret and use data provided by these assessments. Many teachers and students were lacking assessment literacy; therefore, they were unable to adjust their instruction and study habits to increase student performance on summative assessments. This review examines the literature pertaining to assessment literacy for teachers and students, including assessment literacy in mathematics. The review then moves into a discussion on assessments. Through the literature, it was determined that training in assessment literacy for both teachers and students is required to maximize the benefits of using data gathered from formative assessments to adjust instruction and study habits.

Keywords: assessment; assessment literacy; formative assessment; mathematics assessment; summative assessment

A Review of Literature: Assessment Literacy for Mathematics Teachers and Students

In 2001, the legislation known as No Child Left Behind Act (NCLB) was enacted to increase accountability for school districts across the nation, resulting in federally mandated high-stakes testing in reading, science, and mathematics (NCLB, 2002). High-stakes tests may be used to determine what class a student can enroll in or if a student is allowed to graduate. High-stakes tests are also used to determine if schools have met the required adequate yearly progress (AYP) goals as required by the NCLB act. In 2015 a new law was passed called Every Student Succeeds Act (ESSA) that will be implemented in schools starting in the school year 2017-18

(ESSA, 2015). The ESSA retains the feature of annual standardized testing requirements of the 2001 NCLB but shifts the law's federal accountability provisions to the states. Therefore, the propensity of high-stakes tests will still exist.

Raising the standards of learning and achievement is a national priority. National, state and district standards are set to increase the rigor of the courses. Programs for external testing of students' performances are being enhanced (Kaufman, Guerra, & Platt, 2006). Over the years, the assessment community has focused on maximizing the efficiency and accuracy of high-stakes tests. Yet, little attention is paid to assessment as it affects teachers and students in daily classroom use (Lian, Yew, & Meng, 2014; Stiggins, 2007).

Evidence shows that everyday practice of assessment use in classrooms is ridden with problems. These problems include lack of time (Supovitz & Klein, 2003; Wayman & Stringfield, 2006), lack of a technological infrastructure (Chen, Heritage, & Lee, 2005; Lachat & Smith, 2005), and teaching practices that work against use of assessment evidence in an ongoing manner (Ingram, Seashore Louis, & Schroeder, 2004; Supovitz & Klein, 2003; Young, 2006).

For years, an emphasis in educational research has been that of student achievement in mathematics. Assessment is the main area of focus when it comes to measuring student mathematical achievement (Wiliam, Lee, Harrison, & Black, 2004). Not everyone agrees on the tools needed to increase mathematical achievement (Bernhardt, 2006). Both formative and summative assessments have been under the microscope. Formative assessments are designed to inform instruction and study methods to increase performance levels on summative assessments (Popham, 2010). Traditionally, paper-and-pencil versions of formative assessments (i.e. homework assignments, quizzes, chapter tests, and benchmark assessments) have been used to gather data. Recently, Computer Adaptive Tests (CATs) have been added to the list of ways of gathering data from formative assessments. However, many teachers and students do not know what to do with the data obtained from formative assessment (Black, 1993; Lam, 2015; Popham, 2010, 2011). Students' achievement on summative assessments often depends on teachers and students being assessment literate. Assessment literacy refers to the understanding teachers and students have to use data provided by formative assessments to adjust learning experiences in order to gain the required knowledge for success on the summative assessment (Popham, 2008, 2011).

being told it was a number of importance in relation to circles. Its irrationality was also completely overlooked, and I believed that π could be written as the ratio $\frac{22}{7}$ until I was a junior in high school. This error in my thinking followed me to other irrational numbers, such as$\sqrt{2}$ and e. I was never able to shake my misconceptions about irrational numbers because of my initial misunderstandings that were never addressed again in my middle and high school math classes.

In order to avoid the misconceptions and misunderstanding students develop during such approaches to teaching irrational numbers, it is important to introduce the historical development of circles and approximations of π. The history validates the mental struggle students will have in accepting irrational numbers and their properties. Many of these ideas are further illustrated through the use of dynamic geometry software.

A Brief History of π

The definition of a circle is the set of points equidistant from a given point. Historically, circles were "discovered" by almost every civilization because of their natural occurrences (the sun, moon, ripples in the water, plant life, etc.). The next natural step in the exploration of circles was to create them, which begs the question: how much material do I need to create the circumference or area of a circle? Thus began approximations of what we know as π.

A pattern emerged: the circumference of a circle divided by its diameter always resulted in the same number. The nature of this number remained a mystery, but it soon became clear throughout history that π was appearing in this consistent ratio (although it had yet to be discovered with much precision). The formula for the area of a circle was formulated even before a thorough understanding of π was achieved. One of Euclid's lemmas relating to the area of a circle is that "Circles are to one another as the squares on their diameters." His lemma is essentially that these squares are proportional no matter the circle.

To illustrate Euclid's claim, consider the following circles inscribed in squares with the area of the purple circle having area A_1 and diameter d_1, and the area of the green circle having area A_2 and diameter d_2:

The side length of the squares are equal to the diameters of their respective inscribed circles. Writing the areas as proportions according to Euclid's lemma gives:

$$\frac{A_1}{A_2} = \frac{d_1^2}{d_2^2}$$

Utilizing the fact that any diameter is the radius multiplied by two, the proportion can be rewritten as:

A Rational Approach to Irrational Numbers

Kiersten Thorsen, Bryant Middle School

Infinity is a strange concept. In mathematics we encounter infinities that are "countable" and infinities that are "uncountable." Limits approach infinity, and we use the concept of infinity to solve problems, yet it remains not a number but a philosophical concept of mathematics. As abstract a concept as it is, our students can still admit that if they were to try and count the entirety of our set of natural numbers they would never be able to stop.

The concept of any irrational number draws on this sense of "random infinity" that our students have never encountered before. The natural numbers are predictable. The decimal expansions of our rational numbers have a pattern to draw upon. Irrational numbers are unlike any number we have encountered before, and yet simply describing irrational numbers as "not rational" is incredibly unsatisfying. To describe for example an orange, simply saying "an orange is not an apple" tells me nothing concrete about the orange. Teachers have this habit of sometimes describing objects as they are not, and not necessarily as they are.

This doesn't mean rational numbers should not be used as a tool in aiding understanding of irrational numbers, but our logic should never hinge on what something isn't. True understanding comes from comparing the behaviors of each and examining them both together and independently.

Students in the seventh grade are about to explore the first irrational number in the curriculum, π. Our job as teachers seems to be to introduce π in the context of circles and gloss over thousands of years of history in which individuals were supposedly banished and shunned for shattering the assumption that all numbers could be written as a ratio. And yet despite the years of historical struggle, our students are often expected to take in this knowledge as fact with little to no exploration. In recent years education has moved beyond this ideal of the teacher being the deliverer of parcels of knowledge. Now, we allow our students to utilize inquiry in their experience of mathematics. Two numbers being incommensurate is an idea that I anticipate I will be thinking about and struggle with for most of my life, and we owe our students the opportunity to build a foundation for this lifelong journey of thought that does not impede their further explorations.

Building Upon My Own Experience

Although I have not had the opportunity to teach this lesson previously, I would have taught the circumference and area of circles exactly how I was taught. I was taught the formula for the area of a circle in a very direct way; my teacher gave a few contexts in the form of circular objects like wheels and pie dishes, and gave us the formulas to find the areas.

As I experienced this as a student, I gained little to no mathematical understanding about irrational numbers and how the formula actually relates to area. It wasn't until my time in college that I was shown a proof of why the area is πr^2, and I couldn't help but wonder how my understanding of π itself would have changed had I been given any evidence of its appearance in a circle instead of simply

During the past several decades, educators began to discuss the benefits of formative assessment on teachers' instructional decisions and students' studying practices (Popham, 2008). Formative assessment is now an integral part of teaching and learning; however, assessment literacy in not yet an integral part of education.

This review examines the literature pertaining to assessment literacy for teachers and students. The review will then move into a discussion on assessments, including formative and summative assessments.

Assessment Literacy

In order to make assessments worthwhile, Popham (2011) posited that teachers and students needed to become assessment literate. "Assessment literacy consists of an individual's understandings of the fundamental assessment concepts and procedures deemed likely to influence educational decisions" (Popham, 2011, p. 267). According to Havnes (2004), teachers often assumed that it is their teaching that guides the students' learning. However, in practice it is assessment that directs the students' learning and defines what is worth learning (Brown, McInerney, & Liem, 2009; Havnes, 2004).

Havnes (2004) conducted an ethnographic study of a compulsory preparatory course at the University of Oslo. He observed, interviewed, and worked side by side with several students in the course. His main argument was that the assessment structure contributed to the establishment of the learning content, how the teachers taught the course, and the students' learning practices. "Learning is relational. It is relational to assessment, but assessment, again, is relational to other components on the complex system of educational programmes" (Havnes, 2004, p. 171).

In a qualitative study conducted by Lukin, Bandalos, Eckhout, and Mickelson (2004), a group of teachers were trained in assessment literacy through a formal course offered by the University of Nebraska-Lincoln called Nebraska Assessment Cohort, which was an adaption of the assessment literacy training program developed by Stiggins in 2001. The teachers implemented the skills they learned through the course into their classrooms. At the end of one school year, data were collected from one high school to determine the effectiveness of the training in assessment literacy. The researchers used a questionnaire (Classroom Assessment Questionnaire) and a survey (Self-Assessment Development Levels based on Classroom Assessment Quality Rubrics), both developed by Arter and Busick in 2001. Participants answered several open-ended questions about their skill, confidence levels, and changes they had made in their own classroom teaching and assessment practices. The data collected suggested that the assessment literacy learning training had a positive impact on teacher confidence, knowledge, and skill in the area of classroom assessment. There appears to be evidence, while limited, which suggests students also experienced positive outcomes in achievement.

When teachers use data from an assessment, they can better assist students in their learning progressions (Popham, 2008; Shapiro & Gebhardt, 2012) . Data from assessments lead the teacher to develop instruction that is suited to the students' needs. "Assessment-literate educators come to any assessment knowing what they are assessing, why they are doing so, how best to assess the achievement of interest, how to generate sound samples of performance, what can go wrong, and how to prevent these problems before they occur" (Stiggins, 1995, p. 240). It seems likely that the most self-regulating students use formative assessment to improve the quality of their learning progression (Brown et al., 2009). The goal would be to have all students use formative assessment to enhance their education.

Assessment Literacy for Teachers

Assessment literacy for teachers is just as important as assessment literacy for students (Popham, 2008; Stiggins, 1995). Teachers must be assessment literate (Popham, 2011). According to Popham (2011), teachers need to know about the range of assessment strategies so they can maximize the opportunities for gathering evidence. They need to know how to align assessment with instructional goals, and then ensure that inferences drawn from the assessments are of value in aiding the teachers' understanding of where students are with respect to their learning progressions (Heritage, 2007).

Research suggested that teachers need training to interpret data from assessments, and then take that information to adjust instruction to meet the needs of the students (Blink, 2007; Popham, 2008). Observations and interviews to determine that teachers use the knowledge gained from assessment data to inform their instruction have been used to determine the level of assessment literacy (Christman et al., 2009; Ingram et al., 2004; Shepard, Davidson, & Bowman, 2011). One study (DeLuca & Klinger, 2010) used a questionnaire developed by the researchers to determine the level of assessment literacy for all subject

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millions of digits of π. Euler cites an approximation of this constant of over a 120 decimals places in his *Introductio in analysin infinitorum.* With the exponential increase in computing power and more efficient algorithms, the number of digits of π has increased tremendously. The longest decimal expansion of π is 12.1 trillion digits found by Alexander J. Yee and Shigeru Kondo in 2013 (see Yee & Kondo). Their calculation used the Chudnovsky Formula and it took 94 days to compute these digits. This is certainly a far cry from the ancient Babylonians' original approximation of 3 and Archimedes' $3\frac{1}{7} < \pi < 3\frac{10}{71}$. Who knows how far we will progress in another four millennia?

areas in pre-service teachers. The questionnaire focused on the confidence level of differing aspects of several types of assessments. While the pre-service teachers may not have experience in the classroom, they are able to identify their perceived needs, level of knowledge, and sense of readiness pertaining to assessment literacy. The researchers found that pre-service teachers were more confident in their use of assessment *of* learning (summative) than they were with assessment *for* learning (formative). The results supported the need for training in assessment literacy.

In a longitudinal study of nine high schools (Ingram et al., 2004), teachers were expected to use data to assess their own, their colleagues', and their schools' effectiveness in all subject areas and to make improvements. The findings suggested that teachers were willing to use the data to make improvements, but they had significant concerns about the kind of information that was available, how it was to be used, and how it would affect their teaching once they had the information provided by the assessment. These findings are consistent with the characteristics that define assessment literacy. The researcher found that when teachers are trained and supported in becoming assessment literate, greater support for students occurred.

Several studies have found a need for increasing assessment literacy training and understanding teacher conceptions of assessments (DeLuca & Lam, 2014; Lam, 2015; Levy-Vered & Alhija, 2015). Many studies suggest there should be a stronger presence of assessment literacy training in the pre-service teacher education courses and training should carry on to continued support and opportunities for in-service teachers to improve their assessment literacy skills (DeLuca, Klinger, Pyper, & Woods, 2015; Hill, Ell, Grundnoff, & Limbrick, 2014).

Xu and Brown (2016) conducted a review of 100 studies on teacher assessment literacy that concluded in the proposed new conceptual framework of teacher assessment literacy in practice. According to the researchers' proposed framework, assessment literate teachers are continually reflecting on their assessment practices, participate in professional development on how to be assessment literate, and engage in conversation with other professionals about assessment.

Teacher implementation of Assessment

The next two studies show how teachers used their assessment literacy to interpret data and become aware of their students' needs. The teachers then took the knowledge gained through data to inform and adjust instruction to meet those needs.

Wayman and Stringfield (2006) conducted a qualitative study that collected data through focus groups and interviews in three different schools; a pre-kindergarten through grade five elementary school, a large school serving grades five and six, and a middle school grades six though eight. The researchers explored two questions; (a) what facilitates the wide-spread use of examination and learning from student data, and (b) what changes in faculty practice and attitudes resulted from examining and learning from student data. The researchers discovered teachers felt that, along with administrative support, they needed time to learn how to interpret and examine student data. The results indicated teachers were able to use data to go remarkably deep in their examinations of student learning and in their teaching practices. Teacher efficiency was noticeably amplified.

Christman and colleagues (2009) conducted a similar large scale study. They focused on the Philadelphia school district's use of assessments in three key areas: (a) teachers' perception of the assessments; (b) how teachers used the assessments; and (c) how the emphasis on data-driven teaching affected the effectiveness of the exams. Their study utilized multimethods that relied on three sources of data: (a) student achievement and demographic data from 2005-2007, (b) district-wide responses to a teacher survey, and (c) qualitative research from 10 schools during the years 2005-2007. The most important finding from this study was that the success of formative assessments depends on the knowledge and skills of the teachers. Knowledge and skills of the teachers were determined by the amount of training given to the teachers and evidenced in the students' achievement growth. Christman and colleagues conjectured that "data can make problems more visible, but only people can solve them" (p. 65).

Assessment Literacy in Mathematics

Assessment literacy supports mathematics teachers in planning their instruction. As Oláh, Lawrence, and Riggan (2010) discovered, teachers analyzed and used data in two ways: (a) to detect errors, concentrating on whether students got problems correct; and (b) to diagnose those errors, focusing on why students might have gotten certain problems wrong. Some

$$P = \frac{2532}{5000} = 0.5064.$$

And

$$0.5064 = \frac{2(36)}{\pi 45} = \frac{72}{45\pi}$$

Thus, π is approximately equal to $\frac{72}{45(0.5064)}$ or 3.1596, which is about as good as the Egyptian approximation.

In 1901, Mario Lazzarini also carried out Buffon's experiment. This time he used 3408 trials because 3408 = 213 ·16. Lazzarini's experiment is an example of confirmation bias because he kept doing sets of 213 trials until he found his desired outcome. In any case, he was able to approximate π by $\frac{355}{113}$ or 3.14159292, the best rational this approximation of π with less than five digits in the numerator and the denominator.

The approximation of functions using Taylor's series led to the trigonometric methods for approximating π. One such example of this is the Gregory-Leibniz series which is based on the Taylor Series expansion of $f(x) = \tan^{-1}(x)$.

$$\tan^{-1} x = x - {x^3}/_3 + {x^5}/_5 - {x^7}/_7 + {x^9}/_9 - \cdots$$

If we evaluate this at $x = 1$, we get

$$\frac{\pi}{4} = \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$

Or

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots\right) \approx 3.33968253968$$

This result clearly shows the major problem with using trigonometric methods: These series converge so slowly that one needs about 5 billion terms for 10 correct decimal places.

The Gauss-Legendre algorithm provides a much faster technique for finding the digits of π than the trigonometric methods. Carl Friedrich Gauss (1777-1855) and Adrien-Marie Legendre (1752-1833) independently formulated this method. This algorithm involves an iterated process with initial values of $a_0 = 1,\ b_0 = \frac{1}{\sqrt{2}},\ t_0 = \frac{1}{4},\ p_0 = 1$. Each subsequent step involves letting

$$a_{n+1} = \frac{a_n + b_n}{2},\ b_{n+1} = \sqrt{a_n b_n},\ \ t_{n+1} = t_n - p_n(a_n - a_{n+1})^2,\ p_{n+1} = 2p_n.$$

Then

$$\pi \approx \frac{(a_{n+1} + b_{n+1})^2}{4t_{n+1}}$$

We can repeat this process until the desired accuracy of π has been reached. Using a TI-*n*Spire CX, our initial approximation of π is 3.14058. The second iteration gives an approximation of 3.14159 whereas the third iteration is 3.14159265359. The fourth iteration produces the same value. One advantage of this technique over the trigonometric methods is that the number of correct digits doubles with each iteration and can produce 45 million correct digits of π in 25 iterations.

Over a period of approximately four millennia, we traced the approximation of the ratio of the circumference of a circle to its diameter from one decimal digit of accuracy to potentially

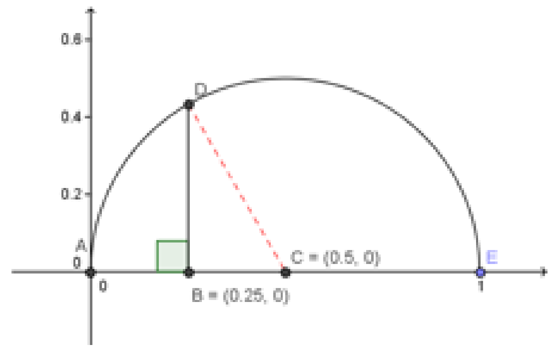


Figure 10

The equation of the semicircle is $y = x^{1/2}(1-x)^{1/2}$. Using the Generalized Binomial Theorem, he expands this equation getting

$$y = x^{1/2}\left(1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4 - \frac{7}{256}x^5 - \frac{21}{1024}x^6 - \cdots\right)$$

This simplifies to

$$y = x^{1/2} - \frac{1}{2}x^{3/2} - \frac{1}{8}x^{5/2} - \frac{1}{16}x^{7/2} - \frac{5}{128}x^{9/2} - \frac{7}{256}x^{11/2} - \frac{21}{1024}x^{13/2} - \cdots$$

To find the area of region ABD, Newton uses his fluxions to `find the integral by the power rule' and evaluates it at $x = \frac{1}{4}$. The area of region ABD is approximately equal to 0.076773123870032. He then finds the area of region ABD geometrically. By construction, the area of region ABD is equal to the difference between area of sector ACD and the area of ΔABD. Since ΔABD is a 30-60-90 triangle, its area is $\frac{\sqrt{3}}{32}$. Since the sector ACD is one-sixth of the entire circle, its area is $\frac{\pi}{24}$. So, the area of region ABD is $\frac{\pi}{24} - \frac{\sqrt{3}}{32}$. So, $\pi = 24(\text{Area (ABD)} + \frac{\sqrt{3}}{32})$, or approximately 3.1415930785574, which is accurate to six decimal places (Dunham).

About one hundred years later, in 1777, Georges-Louis Leclerq, Comte de Buffon (1707-1788), proposed the following problem:

> Suppose we have a floor made of parallel strips of wood, each the same width, and we drop a needle onto the floor. What is the probability that the needle will lie across a line between two strips?

This innocuous, little problem and its ultimate solution lead to a new technique for approximating π. In 1812, Pierre-Simon, Marquis de Laplace, solved Buffon's Needle Experiment using integral calculus. He found that the probability P that the needle will lie across a line between the two strips is $2l/\pi a$, where l is the needle length and a is the distance between the two strips (Burton, pp. 486 – 487).

Buffon's Needle Experiment became the foundation of the Monte Carlo Method of Approximating π. The first to use this technique was Johann Rudolf Wolf (1816-1893). He used a needle 36mm long and the distance between the lines was 45 mm. He tossed the needle 5000 times and it cut the line 2,532 times. So,

teachers looked at the procedures used by students in solving problems, while others focused on underlying mathematical thinking and misconceptions. They found that teachers used their assessment literacy to interpret data from a variety of sources. For example, some teachers reported asking students to explain responses to particular assessment problems, or encouraging students to show their work. Oláh and colleagues also found that teachers' analysis of data led to different types of instructional planning.

Shepard and colleagues (2011) observed and interviewed 30 middle school mathematics teachers in seven districts. They discovered that the amount of assessment literacy possessed by teachers determined the extent to which they were able to use data collected by formative assessments. Teachers' uses of assessment information varied; most frequently they retaught standards or items with the lowest scores. Although many teachers expressed an interest in using assessment results to inform instruction, they reported a minimal amount of professional development in assessment literacy. According to Oláh and colleagues (2010) and Shepard and colleagues (2011), assessment literacy should inform teachers how to interpret, analyze, and use data from formative assessments to adjust instruction.

Assessment Literacy for Students

Not only do teachers need to learn how to use data from assessments, but students also need to learn how to use that information to enhance their achievement (Gibbs & Simpson, 2004; Mac Iver, 1987). Sadler (1998) stated, "Students should be trained in how to interpret feedback, how to make connections between the feedback and the characteristics of the work they produce, and how they can improve their work in the future" (p. 78). Popham's (2008) statement concurs with Sadler that students must begin the process of using assessment data to improve their work by having a "full-scale orientation" (p. 73) on these learning tactics.

Researchers (Hattie, 2012; Hattie, Fisher, & Frey, 2016; Heritage, 2007) suggested that the most powerful single moderator to improve achievement is the feedback students get from assessments. It is not common to have students focus on some form of self-assessment or feedback from assessments (Black & Wiliam, 1998b). Yet, Black and Wiliam (1998a) found in their review of literature a large and consistent positive effect on learning from assessment feedback. This conclusion is echoed in many other researchers' writings since Black and Wiliam conducted their literature review (Black et al., 2003; Popham, 2008; Stiggins, 2007).

Nicol, Thomson, and Breslin (2014) discovered that college students giving and receiving feedback also helped them to engage in a reflective process about their own work. It shifted control of learning into the students' hands.

Participating in assessments without a perceived purpose, combined with a teacher centered approach to instruction, discourages students from fully engaging in their learning (Robinson & Udall, 2006). Although teachers play an important part in educating students, it is a supporting role (Popham, 2008). When students are trained in assessment literacy, adjustments to learning tactics become student-determined instead of teacher-directed (Brown et al., 2009; Popham, 2008). For many students, assessment is not an educational experience, it is a process of "guessing what the teacher wants" (McLaughlin & Simpson, 2004, p. 136). Robinson and Udall (2006) found that students are able to take responsibility for what and how they learn when equipped with the skills to monitor, make judgments, and critically reflect on their performance. These skills include understanding the meaning of the results and feedback from formative assessments and knowing where to look for assistance to fill knowledge gaps when they are discovered in the critical reflection of the results and feedback.

Research has found that providing training to students in assessment literacy can be beneficial (Brookhart, 2001; McDonald & Boud, 2003; Nicol, 2009; Nicol et al., 2014; Smith, Worsfold, Davies, Fisher, & McPhail, 2013). However, "Neither educational researchers nor educational practitioners fully understand how students' thoughts, feeling, and actions ultimately influence their academic success" (Artino & Jones, 2012, p. 174).

Artino and Jones (2012), found that boredom, frustration, and low task value interfere with assessment literacy and can be extremely damaging in the learning context. They surveyed 302 undergraduate U.S. service academy students. Even though these students were skilled at assessment literacy, negative emotions made them less likely to employ adaptive learning strategies.

Brookhart (2001) and McDonald and Boud (2003) considered the impact of training high school students on their performance in assessments. Brookhart conducted a qualitative study by interviewing 50 high school students about specific classroom assessment events. The successful students engaged in self-assessment as a regular ongoing process. They studied for tests, they accepted the challenge of mastering difficult material, and they learned on their own

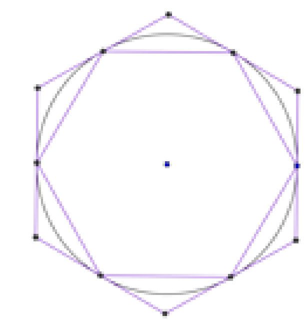


Figure 8

We can easily see that the circumference of the circle is between the perimeters of the inscribed and the circumscribed hexagons. For his second construction, Archimedes bisected the sides of the inscribed hexagon and constructed regular dodecagons (see Figure 9). Again, he measured the perimeters of the dodecagons and knew that the circumference of the circle was between these two numbers.

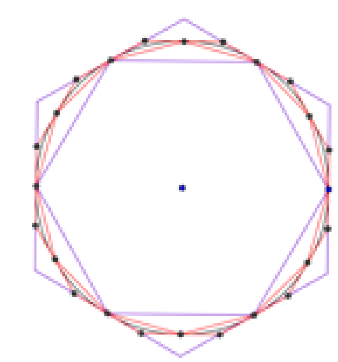


Figure 9

He continued these constructions and calculations by doubling the number of sides of the inscribed and circumscribed polygons until the circle is between inscribed and circumscribed regular 96-gons. As the number of sides increases, the polygons look more and more like circles and the difference between the circumference of the circle and the perimeters of the inscribed and circumscribed polygons becomes smaller and smaller. Archimedes' final approximation of π is $3\,\frac{1}{7} < \pi < 3\,\frac{10}{71}$.

About two millennia later, Sir Isaac Newton (1642 – 1727) approximated π using geometry and his fluxions, as described in *Methodus Fluxionum et Serierum Infinitarum* (1671). He started with a semicircle of radius $\frac{1}{2}$ and centered on $(\frac{1}{2}, 0)$ as shown in Figure 10.

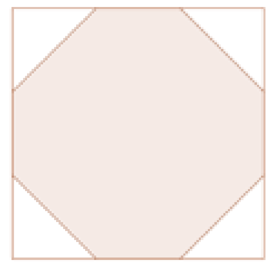


Figure 7

Problem 50 from the Rhind Papyrus describes the Egyptian method for finding the area of a circle:

> Example of a round field of diameter 9 khet. What is its area? Take away $\frac{1}{9}$ of the diameter, namely 1; the remainder is 8. Multiply 8 times 8. It makes 64. Therefore, it contains 64 setat of land.

More generally, the Egyptians found the area of a circle of diameter *d* by

$$(d - {}^{1}/_{9}\ d)^2 = ({}^{8}/_{9}\,d)^2 = {}^{64}/_{81} \cdot d^2.$$

Going back to Problem 48, the area of a circle with diameter 9 is $({}^{8\cdot 9}/_{9})^2 = 64$ which is approximately 63 the area of the octagon in Figure 7.

But what does all of this say about the Egyptians' approximation of π? We find the area of a circle of radius *r* by

$$\pi r^2 = \pi\left({}^{d}/_{2}\right)^2 = \pi \cdot {}^{d^2}/_{4}\,.$$

So,

$$\pi \cdot {}^{d^2}/_{4} \approx {}^{64}/_{81} \cdot d^2.$$

Then

$$\pi = {}^{64}/_{81}\, d^2 \cdot \frac{4}{d^2} = \frac{256}{81} \approx 3.16049$$

So, the Egyptians also knew that π was a little bit bigger than 3 but their approximation was a little more than our approximation.

The greatest mathematician of antiquity, Archimedes of Syracuse (287 B.C.E. to 212 B.C.E.) gave us the well-known approximation of π: ${}^{22}/_{7}$. He found this approximation via a series of constructions of inscribed and circumscribed regular polygons. He started with regular hexagons as shown in Figure 8:

by reading resources. The successful students considered these self-assessments or self-regulations as instances of learning.

McDonald and Boud (2003) directed a quasi-experimental study by training 256 high school students in self-assessment skills. The focus was on constructing, validating, applying, and evaluating criteria to apply to students' work. The researchers surveyed the students to discover their reactions to the training. The survey revealed that training in assessment literacy was a benefit to the students who received it. Both studies, Brookhart (2001) and McDonald and Boud (2003), concluded that students having the ability to self-assess and adjust their planning and study habits were more successful in their careers as students. They were able to plan ahead and prepare adequately for exams.

Nicol (2009), Nicol et al. (2014), and Smith et al. (2013) explored how formative assessment and feedback enabled college students to develop their ability for self-regulated learning. Nicol (2009) and Nicol and colleagues (2014) suggested that assessment literacy helps to develop the skills students need to monitor, judge, and manage their own learning. Smith and colleagues' (2013) quasi-experimental study showed how assessment literacy in students contributed to educational gains. The students who received the intervention in assessment literacy were able to develop ability to judge their own and others' work, which enhanced their learning outcomes.

Assessments

Assessment falls into two different categories: formative and summative. The focus in the next sections will be on these two types of assessments. Formative and summative purposes are different, and thus are usually discussed as two different things (Black, 1998). Formative assessment is designed to provide feedback and to guide in making adjustments in the learning process, both for teachers and for students (Popham, 2008; Schoenfeld, 2015; Stiggins, 2007). Summative assessments measure what was learned after any formative adjustments have been made. "Assessment (formative and summative) is integral to the learning process and something that students 'take part' in rather than something that is 'done to them'" (Robinson & Udall, 2006, p.98).

Formative Assessments

Many prominent researchers in assessment (Brookhart, 2001; Gibbs & Simpson, 2004;

Heritage, 2007; Kaufman et al., 2006; McDonald & Boud, 2003; Oláh et al., 2010; Popham, 2008; Sadler, 2010; Schoenfeld, 2015; Stiggins, 2007; Stiggins & Chappuis, 2006) credit Paul Black and Dylan Wiliam for piquing the current worldwide interest in formative assessment. Black and Wiliam (1998a) defined formative as "encompassing all those activities undertaken by teachers, and/or by their students, which provide information to be used as feedback to modify the teaching and learning activities in which they are engaged" (pp. 7-8). Formative assessment is the process of using data about students' learning to assist teachers to make day-to-day instructional decisions (Black, 1993; Black et al., 2003; Black & Wiliam, 1998a, 2009; Creighton, Tobey, Karnowski, & Fagan, 2015; Heritage & Niemi, 2006; National Research Council, 2001).

A learning environment with formative assessment has many benefits to the student. In terms of mathematics, research has found that even when the teacher's mathematical knowledge was low, the use of formative assessments had an underlying capacity to make sense of students' mathematical understanding and to aid the teacher in responding with the appropriate instruction (Goertz et al., 2009; Hoover, 2009). Popham (2008) boldly stated, "Formative assessment's raison d'être is to improve students' learning" (p. 7).

Many researchers have found that formative assessment data were beneficial in several different levels of learning including college, middle school, and elementary (Diefes-Dux, Zawojewski, Hjalmarson, & Cardella, 2012; Koellner, Colsman, & Risley, 2011; Lachat & Smith, 2005; Nicol et al., 2014; Shepard et al., 2011). Research found feedback from the teachers to be useful to the college students. It helped the students to use their assessment literacy skills when they understood what had been done correctly and what had been done incorrectly from formative assessment feedback (Diefes-Dux et al., 2012). Shepard and colleagues (2011) discovered that formative assessments informed teachers of the concepts that needed to be retaught. However, the teachers wanted more professional development to better train them in assessment literacy in order to know how to use the data from formative feedback. Koellner and colleagues (2011) conducted case studies and found that when teachers use data from the assessments to identify deficient areas in their students' content knowledge; they were able to determine instructional methods that could be used effectively with their students.

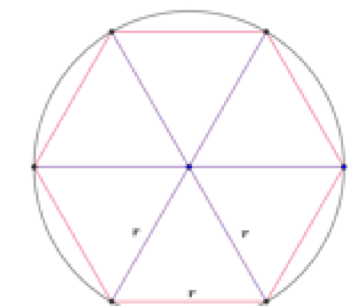


Figure 5

If P is the perimeter of the regular hexagon, $P = 6r$. Also, if C is the circumference of the circle, then $C = 2\pi r$. If we compare the perimeter of the hexagon to the circumference of the circle, we get

$$\frac{P}{C} = \frac{6r}{2\pi r} = \frac{3}{\pi}$$

or $P = \frac{3}{\pi}\,C$. From this tablet we know that the Babylonians knew that $P \approx \frac{24}{25}\,C$. So, we can say that $\frac{3}{\pi} \approx \frac{24}{25}$ or $\pi \approx 3\frac{1}{8}$ (Neugebauer, p. 47). Neugebauer suggests that the Babylonians only used this approximation for π in situations where using 3 was not sufficiently accurate. So, the Babylonians knew that π was greater than 3 but their approximation (3.125) is still a bit less than π.

The Rhind Papyrus, which dates 1650 BCE and was written by the scribe Ahmes, is an important artifact for our understanding of Egyptian mathematics. The papyrus, part of the British Museum collection, contains 85 problems on multiplication, division, applications, and the area of circles. Problems 48 and 50 are concerned with circles. Problem 48 consists of this picture (see Figure 6).

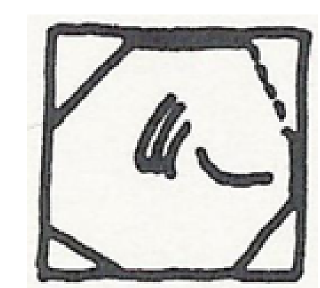


Figure 6

The symbol in the middle is the hieratic symbol for 9 and scholars suggest that the figure hints at a justification for finding the area of a circle. If we let the side of the square be 9 and trisect each side, we can form an octagon as shown in Figure 7. Since the area of each right triangle is $^9/_2$, the area of the octagon is $9^2 - 4(^9/_2) = 81 - 18 = 63$

$\sin(t)$, $d/dt \sin(t) = \cos(t)$. From this point of view, it is no surprise that $z(t) = e^{it} = \cos(t) + i \sin(t)$. At constant speed 1, it takes $t = \tau$ to return to its initial position $z(\tau)=1$, in other words, $e^{i\tau}=1$. And of course, it takes half that time to reach halfway around the unit circle to $z(\tau/2) = z(\pi) = -1$. In other words, $e^{i\tau/2} = e^{i\pi} = -1$.

A History of Approximations of the Ratio of the circumference to its diameter, and a new discovery in the Rhind Papyrus.

Despite this controversy over whether it is more natural to refer of the circumference of a circle to its radius or its diameter, there is still the fundamental question of what is the numerical value of these constants. The better known numerical approximations have focused on Pi, the ratio of circumference to diameter, and we will consider these now. The earliest known approximation of this constant ratio is from the Old Babylonian Period (1800 – 1650 BCE) where the circumference of a circle was found by multiplying its diameter by 3. This factor was also used by the Israelites:

> Then [Hiram of Tyre] made the molten sea; it was made with a circular rim, and measured ten cubits across, five in height, and thirty in circumference (1 Kgs 7:23).

As crude as this approximation was, it was not the only approximation known to the Babylonians. Archaeological excavations in 1936 in Susa (near Shush, Khuzestan province of Iran) uncovered additional mathematical tablets from the Old Babylonian Period. One tablet of particular interest to us provides a list of regular polygons and instructions on how to calculate their approximate area by multiplying the square of their side by associated coefficients (see Table 1).

Equilateral Triangle	$^7/_4$
Square	$^5/_4$
Regular Pentagon	$^5/_3$
Regular Hexagon	$^{21}/_8$
Regular Heptagon	$^{221}/_{60}$

Table 1

For example, to find the area of the regular pentagon, we merely multiply the square of the side of the pentagon by $^5/_3$. So, if a regular pentagon has a side of 4 units, then its area would be $^5/_3 \cdot 16$ or $26\ ^2/_3$ square units. From this tablet, we find a better approximation of the ratio of the circumference to the diameter of a circle by examining the relationship between a circle and an inscribed regular hexagon (see in Figure 5).

Summative Assessments

Summative assessment is an 'overview of previous learning' (Black, 1998, p. 28). Summative assessments can gather evidence over time or at the end of chapters or phases in education (Brookhart, 2001). In Utah, the state summative test used to evaluate students' knowledge is called Student Assessment for Growth and Excellence (SAGE) and was first implemented in 2014 (Utah State Office of Education [USOE], 2013). The SAGE is a high-stakes test that is state mandated due to the pressures of NCLB Act (2002) and will most likely continue to be used with the enactment of ESSA (2015). High-stakes testing in mathematics can have an influential power on the practices of education. Results can affect curriculum decisions, teaching practices, school decisions, and individual students' futures in mathematics (Lester, 2007). Since the SAGE can determine many long lasting decisions, it is important that the students do well on them, while still maintaining a high level of knowledge retention. Genlott and Grönlund (2016) tested 502 students in grades 1 – 3 to see if a formative assessment called Write to Learn (WTL) method of teaching, which uses assessment literacy skills in the form of formative assessment and feedback, would yield improved student results in mathematics and literacy. They found a significant difference in mathematics scores as well as in literacy.

Correlation research studies suggest that when assessment literacy is implemented through the use of proper formative assessments and feedback, the scores on summative assessments, such as the SAGE, improve (Carlson, Borman, & Robinson, 2011; Karpinski, 2010; Keller-Margulis et al., 2008; Nugent, 2009).

Nugent (2009) conducted a study to determine a correlation between a formative assessment and a criterion-referenced summative assessment in middle school mathematics. The results indicated a strong correlation between the formative assessments and the summative assessment. Keller-Margulis and colleagues (2008) found the same strong correlation from 1,477 elementary students' mathematics formative assessments and state-wide end of year assessments. Karpinski (2010) examined the effectiveness of a technology-based formative assessment to predict achievement on a summative state proficiency test. The data showed that students who used technology-based formative assessment to reflect on questions and thoughtfully address the misconceptions (assessment literacy) had a positive correlation to the state test score growth. Although the school districts in these studies collected data from formative assessments, the districts did not have any specific process in place to use these data to make instructional decisions.

Not all research was able to find a predictive ability for summative assessments in mathematics. For example, Donhost (2009), in a study involving students at 86 schools, found no significant difference when he examined the predictive ability of a Computer Adaptive Test (CAT) formative assessment to growth on the state summative assessment for mathematics and language arts. The researcher conducted an analysis of covariance (ANCOVA) comparing the means of the summative assessments of school that used a CAT formative assessments versus schools that did not use the formative assessments. The covariate was the summative assessment scores from a previous year. The researcher also ran a series of t-tests and a linear correlation test to determine whether the reported implementation of data-driven decision making practices (assessment literacy skills) correlated with the summative assessment scores. In both cases the null hypothesis was not rejected with an adjusted R squared = .85 for the mathematics portion of the ANCOVA.

Support for the use of formative assessments to improve summative assessment scores is substantial (Genlott and Grönlund, 2016; Sherman, 2008; Wiliam et al., 2004). However, several studies do not provide sufficient evidence to support the claims made that formative assessment will improve students' summative assessment achievement.

Summary

In order to make assessment worthwhile, teachers and students need to become assessment literate (Popham, 2008, 2011; Sadler, 2010). A significant component of formative assessment data is that teachers and students understand how to use these data. This encompasses the teachers' ability to use data from the formative assessments to adjust instruction as needed and the students' ability to use the feedback from the formative assessments to better their understanding of the material (Popham, 2008).

Assessments are divided into formative and summative assessments. Formative assessments comprise a range of techniques including interviews, observations, homework, and computer-based rapid assessments. The CAT is a form of formative assessments that is aligned to state standards. Summative assessments are the assessments given at the culmination of learning for a chapter, term, or year. Summative assessments are also aligned to

Euler returned to the modern usage, while still recognizing that the quantity was half of another significant quantity, now called τ: "π = the semi-circumference of the circle with radius 1":

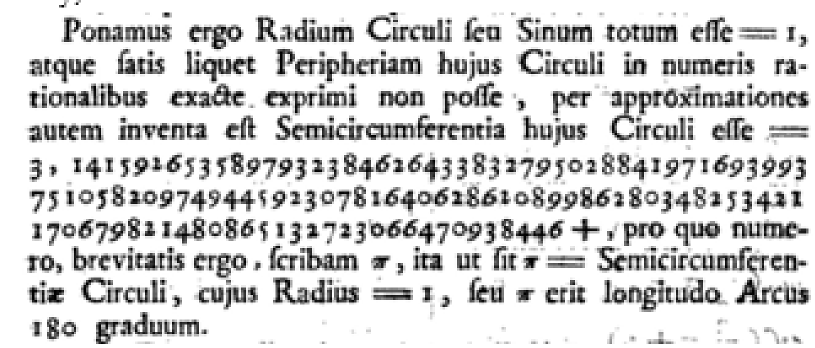
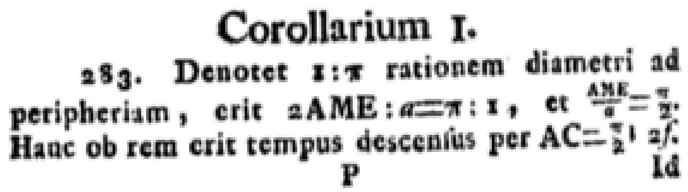

Ponamus ergo Radium Circuli ſeu Sinum totum eſſe = 1, atque ſatis liquet Peripheriam hujus Circuli in numeris rationalibus exacte exprimi non poſſe, per approximationes autem inventa eſt Semicircumferentia hujus Circuli eſſe = 3,14159265358979323846264338327950288419716939937510582097494459230781640628620899862803482534211706798214808651327230664709384460 + , pro quo numero, brevitatis ergo, ſcribam π, ita ut ſit π = Semicircumferentiæ Circuli, cujus Radius = 1, ſeu π erit longitudo Arcus 180 graduum.

Figure 4

So Euler certainly did recognize the natural role of what we now refer to as "τ". He also recognized the number we denote as "π" as "half of something". (N.B. Euler calculated an impressive number of digits with mid-1700s technology!)

The second claim is that π is essential to the wondrousness of the identity $e^{i\pi} + 1 = 0$. This identity is purportedly associated with Euler and involves "five of the most fundamental constants of mathematics." This is dubious on multiple levels. First, this form of the identity is obtained by adding 1 to both sides of the more popular form $e^{it} = -1$. This in turn arises from one of several formulas known as Euler's formula: $e^{it} = \cos(t) + i\sin(t)$, when $t = \pi$. Of course we could add the golden ratio and multiply by the square root of two on both sides to include two more "fundamental constants"! The rearrangement to throw in 1 and 0 already seems *ad absurdem*. What's more, according to the Feb. 2007 issue of Prof. Ed Sandifer's 'How Euler Did It' series, [http://eulerarchive.maa.org/hedi/HEDI-2007-02.pdf] even the association of both the identity and the formula with Euler is questionable. He writes, "Though I agree that it is a beautiful and important result, I am not convinced that we are right to attribute it to Euler" (Sandifer, p. 3).

A geometrical understanding of Euler's formula also supports the primary significance of τ. Many people consider the equality of e^{it} and $\cos(t) + i\sin(t)$ as merely an accidental coincidence of their Taylor series. But a more conceptual explanation is helpful. In calculus, we learn that $f(t) = e^{kt}$ is the function whose rate of change is proportional to its value by k, its proportionality constant, and with an initial value $f(0)=1$. So $z(t) = e^{it}$ is the function whose rate of change is proportional to its value by the proportionality constant i. Calculating $i(x+iy) = -y+ix$ shows that multiplication by i, takes (x,y) to $(-y,x)$, a quarter turn (τ/4 !) counterclockwise rotation. So $z(t) = e^{it}$ is the function whose velocity is always a quarter turn from (or perpendicular to) its displacement and with initial value $z(0)=1$. This is the physical description of uniform circular (harmonic) motion, and the differential equations of the circle: $d/dt\,(x,y) = (-y,x)$: $d/dt \cos(t) = -$



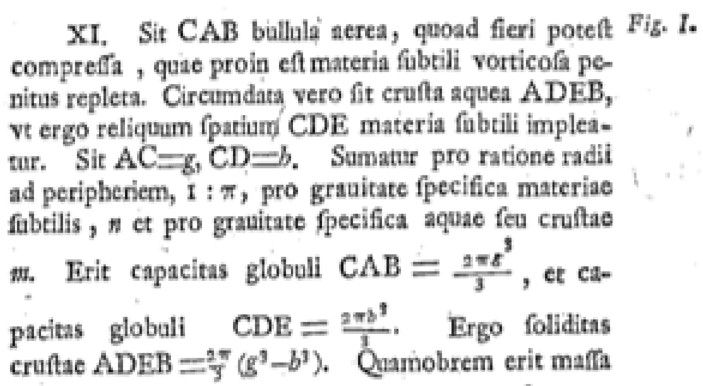
Corollarium I.

283. Denotet 1 : π rationem diametri ad peripheriam, erit 2AME : a = π : 1, et $\frac{AME}{a} = \frac{\pi}{2}$. Hanc ob rem erit tempus descensus per AC = $\frac{\pi}{2}$ √ $a f$.

P Id

Figure 1

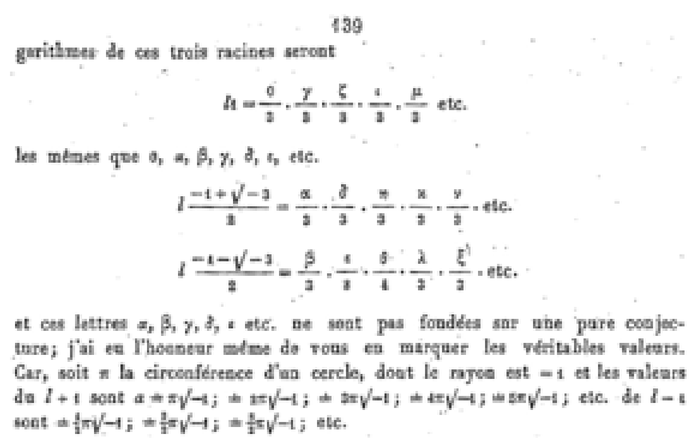
Yet what has been overlooked is his 1727 paper on the theory of air. Euler had already used that symbol for a circle constant "ratio of radius to periphery = 1: π" for the periphery (the first Greek letter of periphery is 'π'):



XI. Sit CAB bullula aerea, quoad fieri potest *Fig. I.* compressa, quae proin est materia subtili vorticosa penitus repleta. Circumdata vero sit crusta aquea ADEB, vt ergo reliquum spatium CDE materia subtili impleatur. Sit AC = g, CD = b. Sumatur pro ratione radii ad peripheriem, 1 : π, pro grauitate specifica materiae subtilis, n et pro grauitate specifica aquae seu crustae m. Erit capacitas globuli CAB = $\frac{2\pi g^3}{3}$, et capacitas globuli CDE = $\frac{2\pi b^3}{3}$. Ergo soliditas crustae ADEB = $\frac{2\pi}{3}(g^3 - b^3)$. Quamobrem erit massa

Figure 2

Even more, he returned to use the symbol π for 6.283… in his 1747 letter to D'Alembert: "…let π be the circumference of a circle of which the radius is = 1…"



139

garithmes de ces trois racines seront

$$l1 = \frac{0}{3}, \frac{\gamma}{3}, \frac{\zeta}{3}, \frac{\iota}{3}, \frac{\mu}{3} \text{ etc.}$$

les mêmes que 0, α, β, γ, δ, ε, etc.

$$l\frac{-1+\sqrt{-3}}{2} = \frac{\alpha}{3}, \frac{\delta}{3}, \frac{\eta}{3}, \frac{\kappa}{3}, \frac{\nu}{3}, \text{etc.}$$

$$l\frac{-1-\sqrt{-3}}{2} = \frac{\beta}{3}, \frac{\epsilon}{3}, \frac{\theta}{4}, \frac{\lambda}{3}, \frac{\xi}{3}, \text{etc.}$$

et ces lettres α, β, γ, δ, ε etc. ne sont pas fondées sur une pure conjecture; j'ai eu l'honneur même de vous en marquer les véritables valeurs. Car, soit π la circonférence d'un cercle, dont le rayon est = 1 et les valeurs du $l+1$ sont $a = \pi\sqrt{-1}$; $= 2\pi\sqrt{-1}$; $= 3\pi\sqrt{-1}$; $= 4\pi\sqrt{-1}$; $= 5\pi\sqrt{-1}$; etc. de $l-1$ sont $= \frac{1}{2}\pi\sqrt{-1}$; $= \frac{3}{2}\pi\sqrt{-1}$; $= \frac{5}{2}\pi\sqrt{-1}$; etc.

Figure 3

If anything, this even better demonstrates Euler's wisdom and flexibility in choosing the clearest and most appropriate notation for the context. In his 1748 *Introductio in analysin infinitorum*.

the state standards. In Utah it is called the SAGE.

In the field of mathematics, formative and summative assessments are given to the students. Data are collected and interpreted by both teachers and students. Teachers and students would then use these data to adjust instruction and study methods.

Formative assessments inform teachers and students of gaps in knowledge. Through the literature, it was determined that training in assessment literacy for both teachers and students is required to maximize the benefits of using data gathered from formative assessments to adjust instruction and study.

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A Brief History of τ: A Useful Alternative to π?

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For millennia, mathematicians have known that the ratio of the circumference of a circle to its diameter is constant. The value of that constant and how it is denoted have a long history. In the first part of this article, we outline the debate surrounding an alternate circle constant: the ratio of the circumference of a circle to its radius. In the second part we look at a history of the approximations of the classical constant.

τ versus π

In recent years, there has been a surprisingly robust discussion of the benefits of using

$$\tau = 2\pi = 6.2831853\ldots$$

as an alternative to $\pi = 3.14159265\ldots$ as a "circle constant", a reference unit for circular measures. A basic observation made in the provocatively titled '*Pi is Wrong!*' (Mathematical Intelligencer, v. 23, no. 3, Springer, 2001) is that one of the earliest and most common contexts in which students and scientists use and encounter π is as the reference for circular measure, and that this is most naturally referred to the length of the full circle of unit radius. In particular, 90 degrees is more naturally a quarter of something, not half. No one has difficulty with converting 15 minutes to a quarter of an hour. So why is this same proportion described in natural radian measure by something over 2 instead of 4? Michael Hartl, a theoretical and computational physicist, made powerful use of the internet and social media to promote the concept and the specific symbol (τ) in the Tau Manifesto. Mathematical physicist Peter Harremoes had simultaneously arrived at the same symbol, τ, for a concept he traces back to the Persian mathematician Jamshid al-Kashi, in the early 15th century. Atomic and molecular physicist Prof. Phil Moriarty contributed the wonderful video Tau Replaces Pi to the excellent Numberphile series and it has been viewed more than two-thirds of a million times! All three physicists expanded on the original arguments and found their own advantages of using τ. (With regard to approximation methods, the hexagon comprised of 6 equilateral triangles of side 1 in the unit circle gives an immediate visual estimate that τ is slightly greater than 6. See Figure 5.)

Not to take this lying down, the π proponents (for example, in the Pi Manifesto) fought back by claiming that Euler first used π to represent this constant and that π is essential to beauty of Euler's identity: $e^{i\pi} + 1 = 0$. Here we wish to briefly address these two historical claims.

The first claim is that no one should question the choice of π = 3.14… as the sole circle constant because the great mathematician Leonhard Euler initially chose it. Who dares to question "Euler: The Master of Us All"? It is true that no one can argue with the brilliance and taste of Euler. And in his "*Mechanica sive motus scientia analytice exposita*"(1736), Euler did write "Denote 1: π the ratio of diameter to periphery"

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I continued teaching in this same manner of geometric exploration with graphing, the Law of Sines (with Ptolemy's triangle), the sine of the sum of two angles (with Ptolemy's triangle), and half-angle and double-angle formulas. Ever mindful of my goal to have students see this subject as "indispensable tools in science, engineering and higher mathematics", I focused on applications of simple harmonics, linear speed, angular velocity, and modeling (such as tide data) more than ship navigation and "how tall is this building" problems. Overall, I was pleased with students' learning, and they reported back that they enjoyed the class.

In summary, students must draw, explore, and think through the mathematics in order to create their own understanding and connect new knowledge with prior knowledge. When textbooks provide a few paragraphs and diagrams explaining a mathematical concept and then go right to practicing with the resulting formulas and procedures, the mathematics is hidden from students. This does not result in meaningful knowledge. Students need experience with mathematical thinking in order to be comfortable with concepts and apply them to new problems. With our geometric exploration of trigonometry, accomplished in a way that allows students to create their own understanding, we transform trigonometry from static ideas of circles and triangles to dynamic mathematics.

A central angle of 1 radian subtends an arc the length of 1 radius. To bring the dynamics of math into this concept, we want to think again of scaling. This figure helps us visualize the scale factor, r. The different circles can be thought of as dilations of the unit circle. Dilation scales lengths by r, but preserves angle measures. Since, by definition, $s = \theta$ in the unit circle, if we scale the circle by r, the length of that segment will be scaled by r as well. Thus, $s = r\theta$.

After this introduction, I ask students what is changing and what stays the same in the figure above. The reason for this is to allow students some time to absorb the new concept and think about the relationship between s, r, and θ rather than just seeing an algorithm to apply. Students shared that if θ stayed the same, the radius and arc length were changing. If θ was changing, then the arc length was changing but the radius remained fixed. We applied $s = r\theta$ to these observations so students could see the effect of changing variables. So now that we've thought about this, why do we care?

When we measure an angle in the unit of degrees, we are measuring how far the terminal side of an angle in standard position rotates through a circle, arbitrarily defined by us (Babylonians) to be $360°$. Since radian measure of an angle gives us the arc length of the circle compared to the radius, then we are measuring the circumference of the circle. What is the arc length when $\theta = 2\pi$? Students quickly decide it's the circumference of the circle, and we revisit $C = 2\pi r$ with perhaps a new appreciation of this equation. If we let $s = C$, and $\theta = 2\pi$, then we have another nice connection to $s = r\theta$.

Let's stay in the unit circle for a minute, which we now know to have a circumference of 2π. From our figure, we see that about 3 radii comprise half of the circumference. A better estimate is 3.14 radii, and about 6.28 radii in the circumference. Converting our irrational circumference number, 2π, to a decimal approximation allows us to comprehend arc lengths in our base ten system. In other words, decimals are quantities we are familiar with. It may be uncomfortable to think of a never-ending, non-repeating decimal length of radius to add to the other six to complete our circumference, but, remember, we know $\sqrt{2}$, an irrational number, is a finite distance.

Why do I go through all of this with students rather than simply using the radius of the unit circle to set up the initial proportion, $2\pi = 360°$? The issue is using an example to convey a procedure. This may lead us to only think of $2\pi = 360°$ as a way to convert between two different measures of an angle. Finally, we care about these points so that it can be clear to students that when they are graphing trigonometric functions, their inputs and outputs are real numbers.

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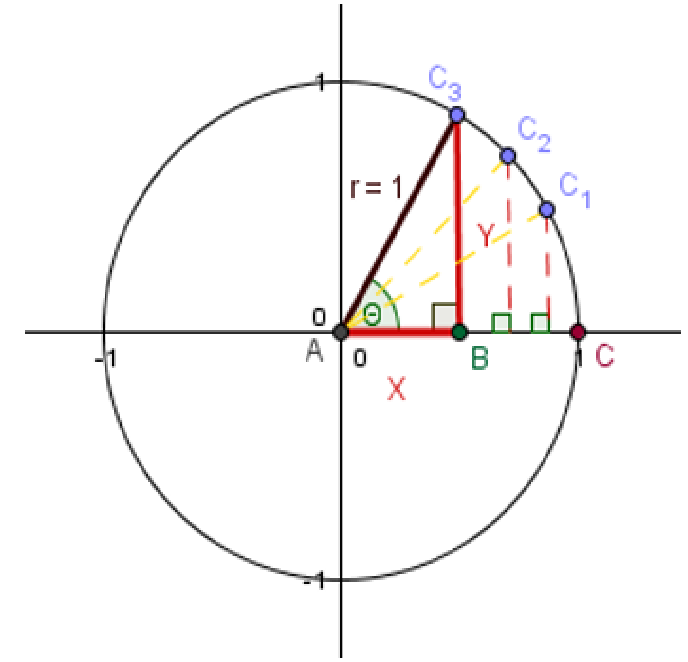
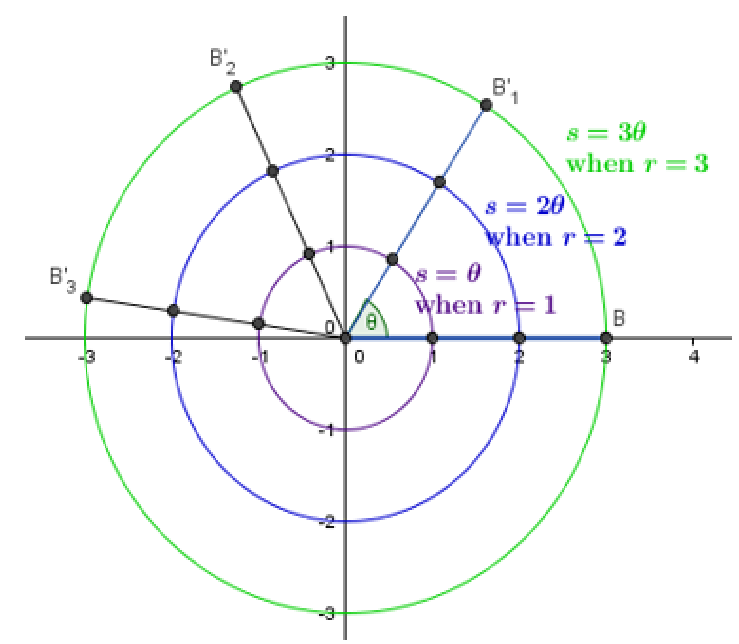
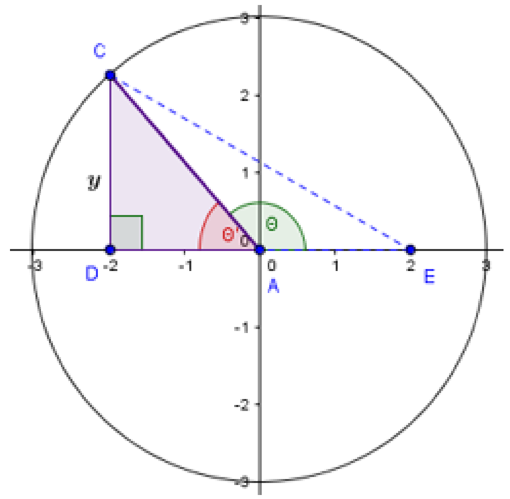


At this point, I decided to introduce radian measure. Measuring θ in degrees only tells us how far around the circle point *P* has rotated. It tells us nothing about the radius or arc length of the circle (the size of the circle). For this, we turn to radians.





This is a nice way for students to start building flexibility with mathematics and how we can represent something (trig ratios, for example) in more than one way. To immediately continue building flexibility, students observe that $sin\,\theta' = \frac{y}{r}$ and y and r are the same in both triangles. It makes sense to define the sine of θ to be the y-coordinate of point C. Aha! We can

eliminate the barrier of $0^0 < \theta < 90^0$ and apply trig ratios to any angle measure of θ. Therefore, we can think of $\sin\theta$ (for example) as the ratio $\frac{opposite}{hypotenuse}$ or $\frac{y}{r}$ or simply as y of the corresponding point on the unit circle.

The interesting piece is talking about the cosine of θ'. Students make the connection that the cosine of θ' is x, and many notice that $-x$ is a reflection across the y-axis of x in ΔACE so they wonder and ask if it matters that x is negative. A negative cosine tells us that θ is obtuse. Aha! Students are satisfied to hear this. They are comfortable making the connection that the sign of the sine and cosine ratios tell us which quadrant we're in. Meaning, given a point on the circle, we know that (x, y) is also $(cos\,\theta\,, sin\,\theta)$ and we then know if θ is acute or obtuse. How about the $\sin 90^0$? Here, we don't even have a triangle! Because $\sin\theta = \frac{y}{r}$ and y and r are the same value, $\sin 90^0 = 1$.

Now they're ready to see trigonometric ratios as functions. Here again is the chance to show that trigonometry incorporates ideas of movement. Going back to this figure, the tangent ratio was discussed. Students quickly made the connection that every x only has one y. After my first class, I was able to bring up the unit circle animation on https://www.desmos.com again.

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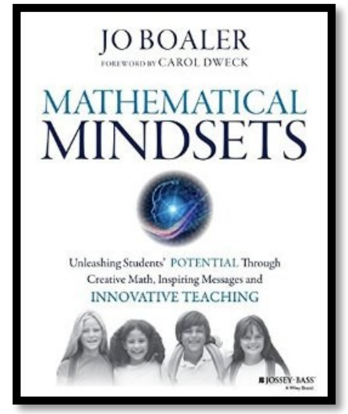
UCTM Recommended Book

Melanie Valentine Durfee, Cedar Middle School



Title: *Mathematical Mindsets: Unleashing Students' Potential Through Creative Math, Inspiring Messages and Innovative Teaching*

Author: Jo Boaler

Publisher: John Wiley & Sons

Year: 2015

Price: $13.58, Amazon

Audience: Mathematics teachers of K-12 students, educators of higher education, and parents

Jo Boaler, Stanford mathematics education professor, authored her seventh book last year regarding mathematics education. Her book, *Mathematical Mindsets,* adds an additional dimension to her previous works. Like some of her prior publications, Boaler's new book includes research to support mindset theory that all students can learn mathematics and that mathematics understanding comes specifically from spending time and effort doing math, not by being endowed with a special "talent for maths." However, in this book, Boaler also gives specific mathematic lessons and rich mathematical tasks for teachers to implement which promote both student confidence in learning mathematical concepts as well as a deepened mathematical understanding. *Mathematical Mindsets* gives teachers specific examples of methods which foster number flexibility, a large collection of classroom-ready, rich mathematical tasks, and suggestions for assessment that promote equity among all students. Boaler weighs in on topics that mathematics teachers are currently debating such as the productive amount of mathematical practice and what computer games increase number sense. Boaler, a proponent of finding the same answer using multiple methods, gives many tasks that teachers can use in their classroom that have multiple entry points and encourage multiple methods, pathways and representations.

Boaler's book is appropriate for K-12 teachers who are interested in promoting mathematical learning in their classroom. The book is also beneficial to parents and others who are

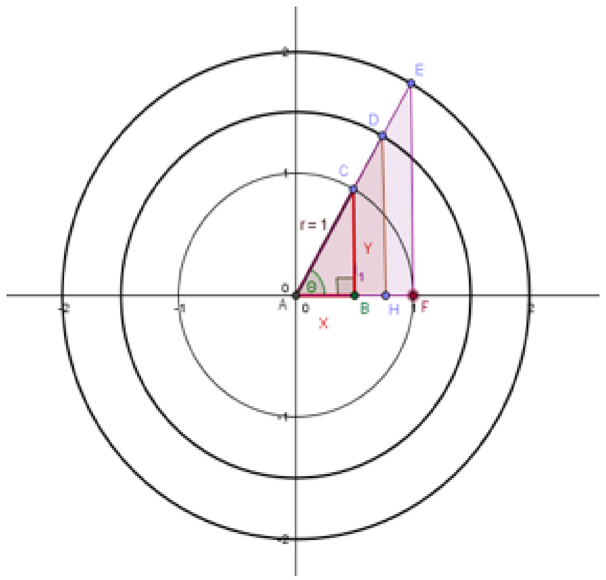
For my third class, I added in the following reference back to history, "We found sine ratios of two angle measures using properties of triangles. Can you imagine finding the sine ratio for every angle measure? How about $\frac{1}{2}$ degree? Remember, Aryabhata, in 500 CE, published a table of sine values and their related shadow ratios for 24 angle measures." An appropriate awed silence followed this last statement.

From the very start, and I say this with some excitement, we have moved away from static geometry and shown trigonometry dynamically. We recall and expand upon students' understanding of similarity by introducing a new concept; for similar right triangles, angle measure is equal to prescribed ratios of side lengths.

We are ready for the next concept. Traditionally, the text has students practice finding trigonometric values of different degrees of θ given a point P on the unit circle. The ordered pair is given in terms of fractions. Students are tasked with simply following a procedure of substitution.

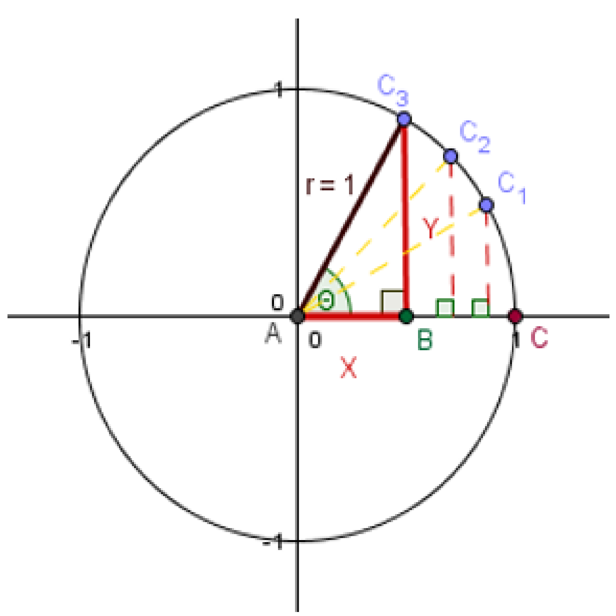
To bring conceptual understanding to this procedure, I had students recall that we can take any right triangle and scale it so that the hypotenuse is 1 unit. By inscribing the triangle in a circle and superimposing this on the coordinate plane, the hypotenuse is also the radius and we can make the following powerful statement: $sin\,\theta = \frac{y}{r} = \frac{y}{1} = y$. This means that $\cos\theta = \frac{x}{r} = x$.

Why is this powerful? After my first class, I found that https://www.desmos.com has a unit circle animation which I shared with students. The animation moves point C along the unit circle forwards and then backwards continuously at a speed I can change. As the animation started from the standard position, I talked about how so far we had learned how trigonometry applied to angle measures of right triangles between 0^0 and 90^0. Once point C passed the y-axis, I paused the animation and had students discover by observation that the $\sin\theta' = \sin\theta$. (Students still have an association of $\sin\theta = \frac{opposite}{hypotenuse}$ and there is no hypotenuse in ΔACE.)



We see that ratios of side lengths of all similar triangles, not just right ones, are the same. Why do we consider the 30-60-90 and the 45-45-90 right triangle relationships? How are they special? This, finally, is where angle measure comes in. The ratios above are uniquely determined by the angle in a right triangle.

For the central angle, this defines $\sin 30^0 = \frac{1}{2}$, $\cos 30^0 = \frac{\sqrt{3}}{2}$ and $\tan 30^0 = \frac{\sqrt{3}}{3}$ and $\sin 45^0 = \frac{\sqrt{2}}{2}$, $\cos 45^0 = \frac{\sqrt{2}}{2}$ and $\tan 45^0 = 1$. In fact, every angle measure between 0^0 and 90^0 has its own ratio of sides.



Teaching Trigonometry: Are we missing the point?

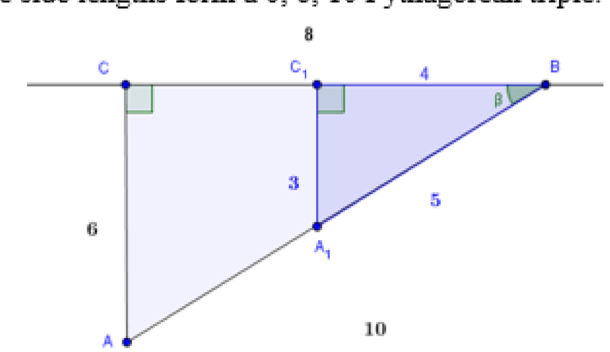
Jennifer K. Sherman, Minot State University, North Dakota

Inspired by George F. Simmons in his book, *Precalculus Mathematics in a Nutshell*, I delved into what we are not teaching in trigonometry. Simmons talks about misconceptions of the subject saying, "What matters most in the subject is not making computations about triangles, but grasping the trigonometric functions as indispensable tools in science, engineering and higher mathematics." This led me to wonder if trigonometry could be thought of as a fluid topic, mathematics in motion, rather than static triangles and circles.

In an effort to find out, I explored how a Glencoe textbook expects students to learn by analyzing the text, recording my opinion of its work and writing my own ideas of how students should learn trigonometry. As a result of this work, I propose that trigonometry can make sense and be itself translated from the world of memorized formulas and procedural calculations to the world of mathematics.

I tried some of my ideas for the first time with a 7-week Trigonometry class in the fall of 2015, for a second time during 7 weeks of a Precalculus class in the fall of 2015, and for a third time in Precalculus spring of 2016. Feeling successful after the first class, I dove in with more commitment for my second and third attempts. Following is a mixed account of how I attempted to allow my students to discover that trigonometry can be thought of dynamically, using geometric concepts, to help make important connections and enhance understanding.

For my first attempt, I began by building on previous knowledge of similar figures encountered in grade eight standards. We can take a textbook example that will require reduction of a fraction and show students similar triangles instead. Let's say, for example, that we have ΔABC, whose side lengths form a 6, 8, 10 Pythagorean triple.



The textbook would have students find the trig ratios for ΔABC as follows:

- $\cos\beta = \frac{8}{10} = \frac{4}{5}$
- $\sin\beta = \frac{6}{10} = \frac{3}{5}$
- $\tan\beta = \frac{6}{8} = \frac{3}{4}$

By drawing similar triangle, $\Delta A'BC'$, with sides 3, 4, 5 and shared angle β and explaining that a scale factor of $\frac{1}{2}$ has been applied to each length of ΔABC, ΔABC~ΔA'BC'. We have the same ratio results, but with an entirely different focus. With similarity, students begin to have a more dynamic, geometric understanding of trigonometric ratios that may otherwise be overlooked.

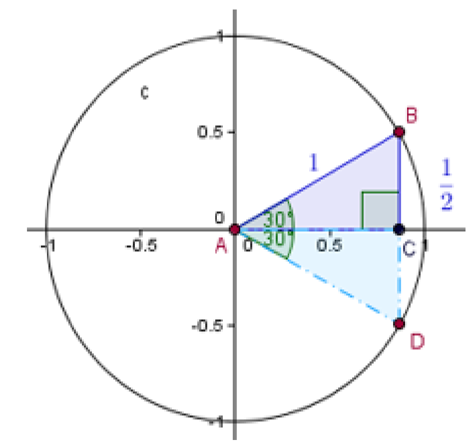
After the success of my first two classes, I knew this geometric understanding was how I wanted to teach, but I hadn't brought any historical perspective to my students yet. I wanted them to experience mathematics as a story that they were a part of. For my third class I began differently, asking, "How did we even discover trigonometry?" The Egyptians didn't know of trigonometry the way we know it today. Thales of Miletus is said to be honored by the King of Egypt for accurately finding the height of a pyramid with the shadow cast from a staff. He placed his staff at the point of the shadow cast by the pyramid, creating two triangles formed by the tangent rays of the sun and showed the ratio of one shadow to the other equaled the ratio of the pyramid height to the staff. This was known as shadow reckoning [3, p.22]. Eratosthenes, in 240 BCE, is remembered for finding the size of the Earth.

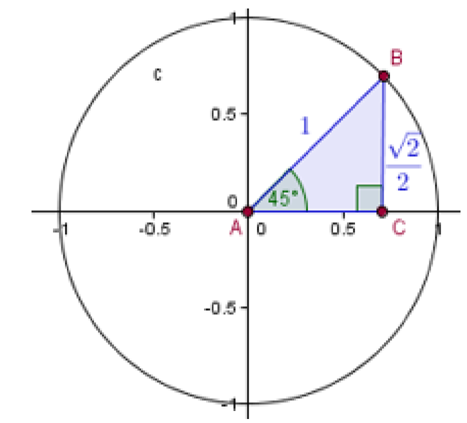
Mathematics continued to develop along astronomical pursuits. Aryabhata, in 500 CE, published a table of sine values and their related shadow ratios. This is where trigonometry was born, and so it is said to have "come out of the shadows". As we can see, it took some time before sides of triangles were related to angle measures. This is what I understand modern trigonometry to be. So, how does trigonometry work?

Now I'm ready to introduce trigonometric ratios with similar triangles, building on the history students just heard about. It is my hope that including historical perspective will change and enrich the way in which they interact and engage with mathematics. We are starting where trigonometry started.

Needing to make up time spent talking about historical perspective, I started with my mathematical structure (a right triangle inscribed in a unit circle superimposed on the coordinate plane) right away in order to show how similarity and the Pythagorean theorem lead us to define trigonometric ratios. I accomplished this task, special right triangle relationships, and historical perspective all in the first day. Previously, this had taken two days (which excluded the story).

To show how similarity and the Pythagorean Theorem lead us to define trigonometric ratios, we bring out the mathematics behind the special right triangle relationships.





Using the Pythagorean Theorem and properties of equilateral and isosceles triangles respectively, we can find side lengths $\overline{AC}$ and $\overline{BC}$ given the radius is 1. We might consider the ratios of sides. Here they are:

	$\frac{BC}{AB}$	$\frac{AC}{AB}$	$\frac{BC}{AC}$
30-60-90 right triangle	$\frac{\frac{1}{2}}{1}$	$\frac{\frac{\sqrt{3}}{2}}{1}$	$\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$
45-45-90 right triangle	$\frac{\frac{\sqrt{2}}{2}}{1}$	$\frac{\frac{\sqrt{2}}{2}}{1}$	$\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$

Interestingly, if we dilate the two triangles from the origin with a scale factor of r, we notice ratios of side lengths are preserved. We can scale any triangle by a factor of r. Can you visualize similar triangles out to infinity?