Utah Mathematics Teacher

Fall/Winter, 2018-2019

Volume 11

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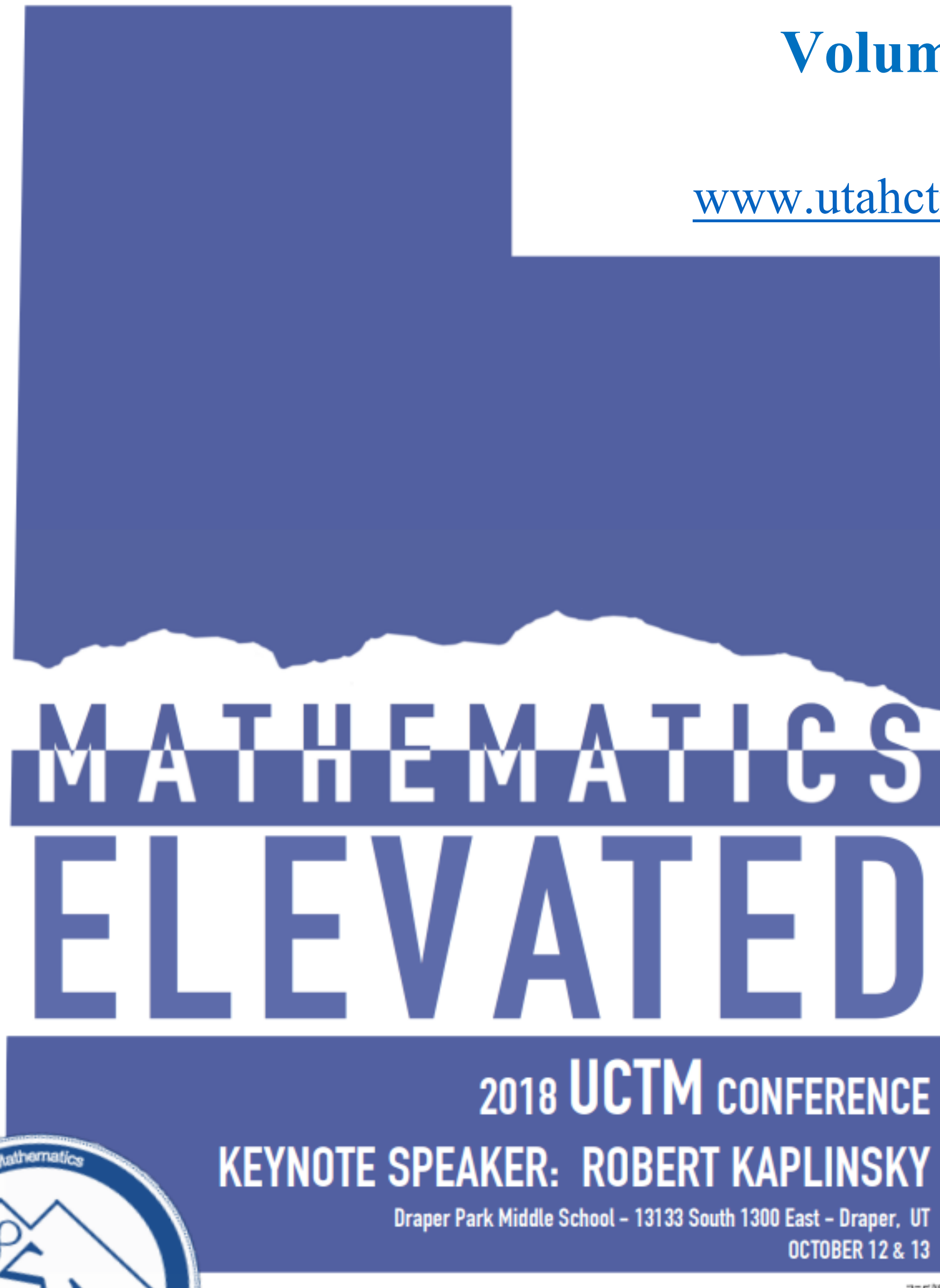

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CALL FOR ARTICLES

The Utah Mathematics Teacher seeks articles on issues of interest to mathematics educators, especially K-12 classroom teachers in Utah. All are encouraged to contribute articles and opinions for any section of the journal. Manuscripts, including tables and figures, should be typed in Microsoft Word and submitted electronically as an e-mail attachment to Christine Walker (Christine.Walker@uvu.edu).

A cover letter containing author's name, address, affiliations, phone, e-mail address and the article's intended audience should be included.

UTAH MATHEMATICS TEACHER

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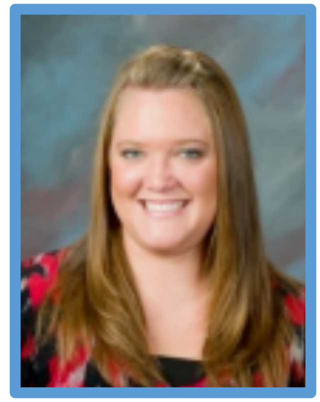
UCTM President's Message

Karen Feld, President, Lehi Junior High



With the start of another school year comes the ability for me to reflect on my teaching practices in the past. This year marks my thirteenth year as a junior high math teacher. I think back on the last twelve years of homework assignments, twelve years of tests, twelve years of parent-teacher conferences and I find myself thinking – what do I need to do differently than I've done before in order to help more students?

Teaching is an ever-changing profession. We change the way we administer a test, grade homework, talk with parents and guardians. However, in all my twelve years of changing and growing while teaching, I have gratefully found that one thing is consistent. My desire and ability to learn. I think as teachers it is in our very nature for us to want to learn and grow so we can try new things and become more effective. We want to pass that desire and love of learning on to our students, hoping that when they get to our stage in life that they will have that desire as well.

We are so lucky to have our National Council of Teachers of Mathematics (NCTM) to help us with our desire to learn and grow as educators. They offer several amazing publications that have made me think about my teaching and grow as an educator. One book that will always be one I come back to is *5 Practices for Orchestrating Productive Mathematics Discussions* by Peg Smith and Mary Kay Stein. This book helped me to know how to reach the goals I had set for myself as a teacher. It was the catalyst in my life that helped me to change the way I interact with my students and help them to think deeply and productively about the mathematics. That's not to say that this change didn't come with lots of tears and frustrations. Change never comes easily. We are creatures of habit and change is always a struggle. However, when we push ourselves to learn more about our profession and then challenge ourselves to change, we find that we are capable of far more than we thought.

So, what has inspired you? What is it that has encouraged you to continue learning and growing as a teacher? How do you want to inspire your students to become life-long learners? I encourage you to think back on your past and how you have become the teacher you are and, more importantly, how you could become the teacher you want to be. There are many great publications from NCTM that can help you learn for the future. There is a 2nd edition of the 5 Practices book that I am very excited to read. I have also found *Principles to Actions* to be a great resource as I think about how to improve my teaching. I hope you have a wonderful school year and continue to be the best teacher and the best student you can be.

Letter from the Editor

Christine Walker, Utah Valley University

I hope each of you have enjoyed a happy, calm and hard-working start to the new academic year. For many of us, we are past the 7-8th week of classes and fall term/semester is in full swing. With it comes a new crop of students bringing an assortment of hopes and dreams. Faculty, staff and administration also bring a special kind of energy to make sure all students succeed in whatever fashion that means.

As I think about the last few weeks, I find myself wondering where the time has gone and how can I find that momentum I had at the beginning of the semester. To jump-start the school year again, I am implementing in Volume 11 a suggestion from the UCTM President-Elect, Amy Kinder. Volume 11 will not only contain new original journal articles but will also include some re-prints of some "Reader's Favorites."

We open the journal with a message from the NCTM President challenging teachers to consider some key questions so that "students might be positioned as mathematically competent in your classroom." Some strategies Dr. Berry suggest are, promoting and valuing students' participation in mathematical discourse and engaging in collaborations aimed at sense making.

This journal features several articles that promote engagement and sense-making, starting with the article titled "Difficulties in Solving Linear Equations," where the author concludes that justifying explanations of how reciprocals are used is a key area of improvement in helping students solve linear equations. As teachers, we know that teaching the unit circle and transformations can be challenging for students, however, several strategies are given in the articles "A Useful Observation about the Unit Circle" and "Graph Transformations by Variable Replacement" that utilizes ideas that facilitate conceptual understanding rather than procedural "rules."

We then turn our attention to two previously published articles that have been "Reader's Favorites" due to the accessible applications in every day classrooms. "Growth Mindset" focuses on learning from known mistakes and utilizing assessment to foster growth and understanding as identified by NCTM. "Using Writing in the Mathematics Classroom" expands on the idea of formative assessment by using verbal and written demonstrations of mathematical understanding to ascertain student thinking.

Above all, we learn in "Turn it Around: Culturally and Linguistically Responsive Teaching," and "Involving Immigrant Parents in the Mathematics Education of Their Children," that by learning about our students, their interests and experiences by involving immigrant parents in the process, teachers can reduce mathematical misunderstandings and help students love math regardless of the students' language and culture.

We close the journal with two different proofs for the limit of a product as an interesting way to contrast the typical proof of the Product Rule for Limits of Functions by considering student reasoning.

I hope you enjoy this journal as much as I had in collecting, reading, reviewing, and discussing the articles with the review committee. In addition, as always, please consider submitting your own articles, or serving as a reviewer for future journal articles.

A very special thanks to Amy Kinder who did the production for the online journal.

Note: Any mistakes are the sole responsibility of the editor and will be remedied in the online journal. Please send corrections to Christine.walker@uvu.edu.

Letter from NCTM President, Positioning Students as Mathematically Competent

Robert Berry, NCTM President

Robert Q. Berry III

NCTM President, 2018-2020

President's Messages

Promoting and valuing students' participation in mathematical discourse—sharing their reasoning; creating, critiquing, and revising arguments; and engaging in collaborations aimed at making sense of and using mathematical ideas—is a way of positioning them as being mathematically competent. In order to ensure that each and every student not only understands and can make use of foundational mathematics concepts and relationships but also comes to experience the joy, wonder, and beauty of mathematics, we must position each and every student as mathematically competent. This requires creating classroom structures—norms and routines—that support students to take risks to engage in discourse and to see themselves as capable and worthy of being heard. In doing so, students' mathematical identities are connected to their participation in a set of productive practices and processes of doing mathematics. Aguirre, Mayfield-Ingram, and Martin (2013) define mathematical identity as "the dispositions and deeply held beliefs that students develop about their ability to participate and perform effectively in mathematical contexts and to use mathematics in powerful ways across the contexts of their lives" (p. 14).

However, in too many mathematics classrooms mathematical competence is assigned solely on the basis of quickness and correctness, giving the mistaken impression that only some students are "good at math." This creates an environment where students' mathematical reasoning goes unexamined and unvalued; consequently, little is known about how they make sense of mathematics, how they use their mathematical understanding in developing solutions, and why their solutions do or do not make sense. Correct answers matter but not as indicators of who is able to do mathematics. Engaging in mathematical discourse is essential for developing mathematical identity and should be recognized as a better indicator of mathematical competence. In what ways must our classrooms and lessons change to promote positive mathematical identities for each and every student?

To get an understanding of positioning students as competent, I invite you to watch Video Clip One from the Bike and Truck Task found in NCTM's *Actions Toolkit*. The video clip is drawn from a high school algebra 1 class but practices modeled in this lesson have strong relevance across all grade bands. To set up the discussion the teacher, Ms. Shackelford, invented a fictional student, Chris, to help her students focus and clarify their thinking about the graphical representation of the position of the truck as a function of time. After watching the video, think about the following questions:

- What norms and routines must have been established and practiced to allow students to engage in the sort of mathematical discourse that positioned each of them as mathematically competent?
- How do the forms of participation move the students forward in their thinking about the mathematics?
- What would happen to the students' respective mathematical identities if this same task unfolded in a different classroom in which Jacobi is told immediately his reasoning is incorrect and Charles is told he is correct?

The video clip is an illustration of how Ms. Shackelford engaged students in reasoning and sense making through a routine of listening to and critiquing others' reasoning. Ms. Shackelford positioned Jacobi (yellow shirt) and Charles (maroon shirt) as capable contributors to mathematical discussion. Jacobi's reasoning did not fit the graphical representation of the truck but he was highly participatory and was able to interact with Charles, whose reasoning did fit the graphical representation of the truck. In the clip we see Jacobi and Charles engaged in public sense-making by sharing their mathematical thinking with their peers and Ms. Shackelford. By publicly making the interaction between Jacobi and Charles worthwhile, Ms. Shackelford positions both students as having mathematical competence through their participation. Their ideas were welcomed and used to build mathematical understanding. When students share and value their mathematical ideas through processes of mathematical discourse, they move away from mathematics competence as producing correct answers quickly and toward mathematics competence as participatory.

Ms. Shackelford conducted this lesson in April of the academic year, and it appeared that the social norm in her classroom had been firmly established and that her students were well aware of, and comfortable with, her expectations that they would explain their thinking, respectfully critique others' reasoning, and make mathematical connections. The lesson in Ms. Shackelford's classroom also modeled intellectual authority as being shared between the teacher and students. As students author ideas, decide and justify whether particular ideas are reasonable, and press one another for explanations, they take on forms of intellectual authority that support collaborative mathematics teaching and learning (Langer-Osuna, 2017).

Positioning students as mathematically competent must happen with clarity and consistency to have a long-lasting positive impact on their mathematical identities (Munson, 2018). The questions below are a start for reflecting on how students might be positioned as mathematically competent in your classroom.

- How do I create classroom norms and routines that support students to take risks to engage in mathematical discourse?
- In what ways are students' mathematical ideas shared and valued?
- How do my teaching practices communicate to each and every student that their ideas matter?
- In what ways is intellectual authority distributed in my classroom?
- How do my teaching practices use students' ideas to guide them to important mathematical insights and understandings?

I encourage you to use the questions to reflect on your classroom and teaching practices. Please share your successes and challenges on MyNCTM.org.

Robert Q. Berry, III
NCTM President

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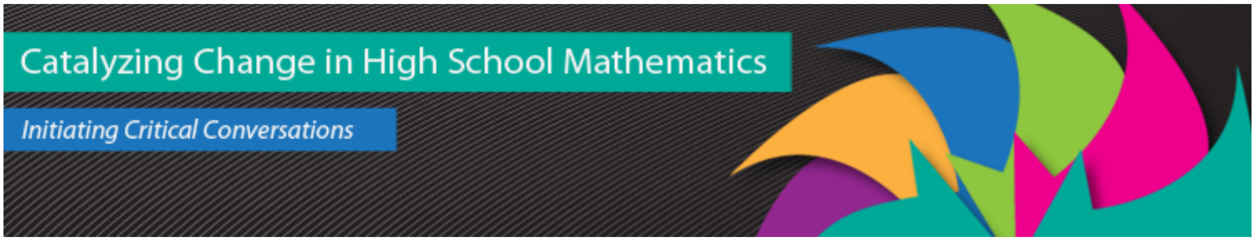
Aguirre, Julia, Karen Mayfield-Ingram, and Danny Martin. *The impact of identity in K-8 mathematics: Rethinking equity-based practices*. Reston, VA: National Council of Teachers of Mathematics, 2013.

Langer-Osuna, Jennifer M. "Authority, identity, and collaborative mathematics." *Journal for Research in Mathematics Education* 48.3 (2017): 237-247.

Munson, Jen. "Two Instructional Moves to Promote Student Competence." *Teaching Children Mathematics* 24.4 (2018): 244-249



Create Positive Change for High School Mathematics

Today's students face a future where there is an increasing need for mathematical skills in the workplace. As a high school teacher, leader, administrator, or counselor, part of your profession involves helping ensure that students are prepared for both personal and professional success.

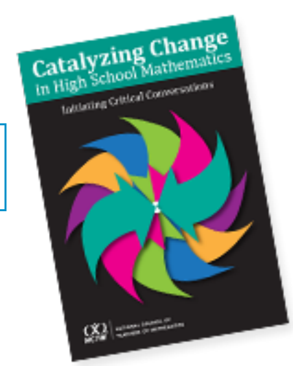
NCTM's new publication, *Catalyzing Change in High School Mathematics: Initiating Critical Conversations*, is a must-read for anyone who's involved in high school mathematics education.

Themes include:

- Broadening the purposes for teaching high school mathematics beyond a focus on college and career readiness
- Dismantling structural obstacles that stand in the way of mathematics working for each and every student
- Implementing equitable instructional practices
- Identifying essential concepts that all high school students should learn and understand at a deep level
- Organizing the high school curriculum around these essential concepts to support students' future personal and professional goals
- Providing key recommendations and next steps for key audiences

Catalyzing Change engages all individuals with a stake in high school mathematics to catalyze critical conversations across groups.

Order your copy today!



Catalyzing Change Executive Summary

Catalyzing Change Infographic

NCTM News Release

Author Talk Webinar

Catalyzing Change in High School Mathematics: Initiating Critical Conversations

Speaker: Matt Larson
May 16, 2018

This webinar provides an overview of *Catalyzing Change* and initiates critical conversations centering on the following serious challenges: explicitly broadening the purposes for teaching high school mathematics beyond a focus on college and career readiness; dismantling structural obstacles that stand in the way of mathematics working for each and every student; implementing equitable instructional practices; identifying Essential Concepts that all high school students should learn and understand at a deep level; and organizing the high school curriculum around these Essential Concepts in order to support students' future personal and professional goals. *Catalyzing Change* is written to engage all individuals with a stake in high school mathematics in the serious conversations that must take place to bring about and give support to necessary changes in high school mathematics.

View the Recording

Catalyzing Change Creating Conversations in the Media

High Schoolers Should Take 4 Years of Leaner, More Relevant Math
Apr 25, 2018 l Ed Week

Mathematics Education: Initiating Critical Conversations
May 1, 2018 l Ed Week

Unlocking STEM Pathways for All Students Which policies open doors for students to STEM—and which slam them shut?
May 22, 2018 l Ed Week

Retooled Courses Help Students Avoid a Remedial-Math Roadblock to College
May 22, 2018 l Ed Week

Calculus Is the Peak of High School Math. Maybe It's Time to Change That
May 22, 2018 l Ed Week

Don't Track Algebra
July 17, 2018 l Ed Week

Difficulties in Solving Linear Equations That the Contain Fractions

Thomas Mgonja, Utah State University

Studies show that students' performance in solving linear equations drops significantly when the equations contain fractions, compared to solving equations with integers only. Linsell (2008) found that the success rate for solving equations dropped from 84% to 25% when fractions replaced integers in the equations. The performance dropped even when the equations were of similar form (e.g., n+46=113, 2 + n/4 = 8). While the low performance of the students' solving equations containing fractions is significant, precisely where their difficulties lie needs investigation. There is little research on exactly what difficulties students experience when solving linear equations that contain fractions. Most of the research that addresses the issue of fractions in equation solving focuses either on the computation of fractions (Brown & Quinn, 2006) or on the development of different problem-solving strategies such as inverse, cover up, or transformation (Linsell, 2009).

The purpose of this study was to examine the difficulties fractions contribute to the process of solving linear equations. The aim was to try to understand the significant drop in performance from solving equations containing integers to solving equations containing fractions. We hypothesized that fractions may extend the current difficulties in solving linear equations to a new dimension or fractions may create new difficulties that are different from previous solution processes.

Literature Review

The literature review focuses on common difficulties found in solving linear equations that contain integers. We believe that we may encounter these same difficulties when solving equations with fractions. It is even possible that fractions will bring these common difficulties to another level of complication. The following paragraphs summarize symbolic, procedural, strategic, and conceptual difficulties students encountered in solving linear equations containing integers.

Operations on Variables

Studies have shown that students have difficulty working with variables. One of the common difficulties is that students are unable to see a variable as a number (Stacey & MacGregor, 1997). It was found that these students misinterpreted the variable as a label for an object or view it as only equivalent to a single number like 1 (e.g., 9 + x = 10). Another difficulty is that students are unable to group or combine variable terms (Herscovics & Linchevski, 1994). These students either treated variables as objects that could not be manipulated like numbers, or they could not differentiate them from numbers, resulting in them adding variable terms and numbers together (e.g., $2x+3=5x$). In fact, Herscovics et al. (1994) showed that 38% of a sample of 3000 students simplified $2x+3$ to either $5x$ or 5.

Equals Sign

The equals sign in an equation does not imply "perform the operation" as defined in arithmetic. It is the relational meaning of equality of two expressions. However, this relational meaning of the equals sign is not always emphasized in teaching and not always evident in students' understanding (Knuth, Stephens, McNeil, & Alibali, 2006). More than half of middle school students in the U.S. understood the equals sign as operational rather than relational (Knuth's et al., 2006). Some methods have been proposed to help students understand the relational meaning of the equals sign, using arithmetic identities such as those found in the study by Herscovics and Kieran (1980).

Equation Transformation

Students are often able to perform equation transformations without understanding that the rationale behind the transformations is based on equivalent equations (Steinberg, Sleeman & Ktorza, 1990). Indeed, students often misuse these transformations (Davis and Cooney, 1977) by, for example, adding or subtracting a number to only one side of an equation rather than doing the same operation to both sides of the equation. Other students might multiply or divide some terms, rather than every term, when multiplying or dividing both sides of an equation (e.g., $2x=4x+6$ becomes $x=2x+6$). These transformation errors are related to the application of the additive and multiplicative properties of equality. Theoretically, the identical relationship between the right side and left side of an equation justifies (or defines) the two properties of equality. Equivalent equations that are transformed by means of these two properties make equation transformation valid.

Strategies in Solving Linear Equations

Strategies for solving linear equations include "trial & error," "undo," "isolate the variable," the standard algorithm, and other advanced strategies. In order for students to understand the meaning behind equation solving, teachers tend to use "trial-and-error/guess-and-check" (Bernard & Cohen, 1988; Linsell 2009) and "undo/inverse" (Benard & Cohen, 1988; Linsell 2009). Teachers, particularly, use trial-and-error to demonstrate that the idea is to find a value for the variable that would make the equation true. This is done by guessing and testing the values in the equation. The "undo" method helps students understand operations on the variable and the process of getting back to the original value of the variable. For example, if "x is multiplied by 2", then the "undo" step would be to "divide x by 2". By undoing, the reverse process leads to the original value of the variable x. Both the "trial & error" and "undo" methods emphasize the idea of solving an equation for finding the value of the variable.

Studies show the two strategies "trial-and-error" and "undoing" are difficult to perform when there are variables on both sides of an equation (Linsell, 2009). In such cases, another strategy named equivalent equations (or equation transformation) is suggested (Benard & Cohen, 1988; Linsell, 2009). With this strategy, equation solving is achieved by using the additive and multiplicative properties of equality to attain equivalent equations. The original equation is transformed to equivalent equations until reaching the form "x = a", where "a" is a constant. At this point, it is said that the variable has been isolated. Star and Seifert (2006) show that this strategy can be further developed into a standard algorithm, which is (1) Removing parentheses by using the distributive property (2) Changing the equation into a standard form ($ax + b = cx + d$) by combining like terms on each side (3) Getting variables on one side and constants on the other side (4) Dividing both sides by the coefficient of the variable.

Advanced strategies are suggested after students master the simpler strategies mentioned above for solving equations. Advanced strategies include "change of variable" (Star, 2007) where operations are applied to similar expressions such as $2(x+2) + 3(x+2) = 10$ being transformed to $5(x+2) = 10$. Another suggested strategy is "clearing fractions" when a student encounters an equation containing fractions. This is achieved by multiplying both sides of an equation by the lowest common denominator of the fractions (e.g., multiplying both sides by 6 when solving the equation $\frac{x}{3} + \frac{5}{6} = 3x$). Finally, students may also use the strategy of "dividing both sides" of an equation by a number that is a factor of every term in the equation (e.g., dividing both sides by 5 to solve the equation $5x + 10(x+3) = 25$). It is believed that students use these advanced strategies to make it either easier or faster to solve equations.

Relationship between the Literature and the Proposed Study

According to the literature review, fractions may contribute to the dynamics of solving equations by increasing difficulty in four phases. The first phase is that variables combined with fractions may complicate students' operations. For example, $\frac{1}{7} + \frac{x}{7} = \frac{2}{7}$ where the variable is treated as a label or equivalent to 1 (Stacey & MacGregor, 1997) or $\frac{2}{7}x + 3 = \frac{5}{7}x$ where a student fails to group or combine variable terms (Herscovics et al., 1994). The second phase is having to do with the equals sign and fractions. Fractions may confuse students to interpret the equals sign as a request to "perform the operation" rather than the equivalent relationship between the expression of each side of an equation (Knuth et al., 2006). The third phase concerns the difficulties fractions impose on transforming equations. Fractions may confuse students when using the additive or multiplicative properties of equality, thus leading them to incorrect equation transformations. This postulation of transformational challenges is inspired by the work of Steinberg et al. (1990) and Davis and Cooney (1977). The fourth phase lies on the difficulties fractions may pose in performing problem-solving strategies. Fractions could distract students from carrying out problem-solving strategies. For example, their failure to perform basic fraction operations may cause them to get lost in the middle of the "isolating the variable" strategy or other strategies that were previously mentioned. In order to understand how fractions, affect students in the four phases, a qualitative study that includes a problem-solving task and a one-to-one interview is employed. The target population is students who just finished a pre-algebra course that covered solving one-variable linear equations containing fractions.

To identify phase 1 difficulties, a variable associated with a fraction (e.g., $\frac{2}{5}x$, $\frac{3x}{7}$) was used. To identify phase 2 and 3 difficulties, operations and equation transformations with both integers and fractions involved were used (e.g., $\frac{2}{5}x+4=10$). The aim of phase 2 and 3 was to investigate and differentiate the impacts of integers and fractions on the difficulties in solving equations. To identify phase 4 difficulties, an equation containing only fractions (e.g., $\frac{3x}{7}+\frac{1}{6}=\frac{1}{3}$) was used to investigate how fractions impact the implementation of problem-solving strategies (e.g., isolating the variable) from the beginning to the end.

Method

This study employs an exploratory research design that aims to identify and describe the difficulties students experience when solving linear equations that contain fractions. Data were collected through interviews and were analyzed with open and axial coding (Miles & Huberman, 1994; Strauss & Corbin, 1998). The specific research questions for the study were: (1) What are the mistakes students make in solving linear equations with fractions? (2) What are the difficulties students run into when solving linear equations containing fractions? It is only by identifying these mistakes and difficulties that we will be able to write an informed recommendation on how to improve performances in equation solving when fractions are involved.

Participants

The participants in this study were six students (three freshmen, one sophomore, one junior, and one senior) who had just completed a fall semester university developmental mathematics course in pre-algebra that taught solving linear equations in one variable that contain fractions. The six students were recruited at a large open enrollment university in the Western United States.

Procedures

Each student participated in a 20-minute one-to-one interview session with a researcher. The students completed ten problems on a worksheet and were interviewed while working on problems #1, #2, #6, and #10. Each interview was audio recorded. Calculators were not allowed; neither were written nor support material. Students were asked to show all their work on the worksheet.

Instrument

The participants were asked to solve 10 mathematics problems on a worksheet, and they were asked several questions by the interviewer before or after solving problems #1, #2, #6, and #10. Students were not asked any questions when solving problems #3, #4, #5, #7, #8, and #9.

The first two problems on the worksheet were simple one-variable linear equations containing only integers. These were followed by three problems that were not related to linear equations. Problems #6 and #10 were two linear equations containing fractions. Problems #7, #8, and #9 were not related to linear equations. Table 1 shows the 10 problems.

Table 1

The Problem-Solving Worksheet

Problem #	Problem	Problem #	Problem
1	Solve: $2x+1=5$	6	Solve: $\frac{2}{5}x+4=10$
2	Solve: $3x+2=9$	7	Add: $-18+6$
3	Evaluate: $\sqrt{36}$	8	Simplify: $2x-5+x$
4	Evaluate: 2×0.5	9	Evaluate: $-20\div -2$
5	Evaluate: $-2+(-3)-(-1)$	10	Solve: $\frac{3x}{7}+\frac{1}{6}=\frac{1}{3}$

The purpose of problems #1 and #2 were to help students recall and become familiar with the method of "isolating the variable" for solving one-variable linear equations. The interviewer provided instructional assistance to the interviewees on the two problems, as needed. After they successfully solved problems #1 and #2, each interviewee was prompted to solve questions #3 to #10. The interviewer reminded the interviewees of the solution method if they were not confident in any step of the solution procedure.

The purpose of problems #3 to #5 and #7 to #9 was to prevent students from memorizing the problem-solving procedures of solving the previous problem. It was hoped that students were not simply replicating the solution method of the last problem, but rather they were redeveloping their plans and thought processes before attempting each subsequent problem. These problems also served to separate the different levels of fractions involved in an equation.

The main difference between problems #6 and #10 was that problem #6 involved only division of fractions and problem #10 involved both the division and subtraction of fractions. Another feature of problem #10 was that the variable x was placed on the numerator of a fraction. The two problems were designed to capture the four phases that fractions may contribute to the dynamics of the equation solving process. For problems #6 and #10, the interviewer used three kinds of questions to investigate the difficulties the interviewees encountered when solving the two linear equations. Table 2 lists examples of the three types of questions used by the interviewer.

Table 2

Interview Questions

Question Type	Example Interview Questions
Planning	What is your plan to solve this equation? What were you hoping to achieve by doing your first step? Could you tell me why you are doing your first step?
Confidence	Is there any step that you are not sure/confident about? Could you tell me if there is any step in there that made it difficult?

Question Type	Example Interview Questions
Walking through	Could you walk me through it? Could you walk me through it as you are solving it?

Data Analysis

Researchers transcribed the audio recordings of the participants' word-by-word. One of the researchers completed the transcription and the other author proofread the transcription. The transcripts were analyzed by open and axial coding (Miles & Huberman, 1994; Strauss & Corbin, 1998) of the participants' language based on the standard algorithm (i.e., isolating the variable) for solving linear equations. In particular, the researchers identified language that focused on: (1) combine like terms; (2) subtract the constant term (on the side with the variable) from both sides; (3) subtract the variable term (on the side with the constant term) from both sides; and (4) divide both sides by the coefficient of the variable. Students' difficulties were probed based on the four stages of the standard algorithm. Participants' difficulties were identified and analyzed based on the following four questions: (1) What are the difficulties? (2) When do the difficulties occur during the problem-solving steps? (3) How are the difficulties related to the other steps (e.g., the connection to the previous or the next step in the problem-solving process)? (4) Why are the difficulties occurring (reasons given by the interviewees)?

Results

The results are organized by problems #6 and #10. For problem #6, the findings are presented in terms of the traumatic reactions students expressed, their algorithmic replications with no confidence, and the concerns they had about division of fractions. For problem #10, the findings are arranged in terms of those students who were unable to begin the problem, those whose operations were overriding strategy, those who were mixed-up due to the position of the variable *x*, and those who were confused in multiple ways.

Five out of the six students correctly solved problem #6. However, only one student solved problem #10 correctly. The details are shown in Table 3.

Table 3

Students' Problem-Solving Results for Six Participants

Participant Number	001	002	003	004	005	006
Problem #6	Incorrect	Correct	Correct	Correct	Correct	Correct
Problem #10	Incorrect	Incorrect	Correct	Incorrect	Incorrect	Incorrect

Problem #6: Difficulties in removing a fraction from the variable

In problem #6, students were asked to solve the equation $\frac{2}{5}x + 4 = 10$. Five of the six students were able to solve for *x* in this problem. However, they encountered some difficulties and expressed the following concerns: traumatic reaction, algorithmic replication with no confidence, and the division of fractions. The students who encountered the three difficulties are shown in Table 4 below.

Table 4

Students and Difficulties about Problem #6

Difficulty/Concern	Frequency	Students
Traumatic Reaction	1	001
Algorithmic Replication	5	001, 002, 003, 004, 006
Division of Fractions	4	001, 002, 003, 004

Traumatic reaction. Student 001 had a traumatic reaction to fractions. The student said, "*So I would subtract…and then from there, I am drawing a blank...(2/5)x equals 6…cause this is a fraction…I hate fractions.*" During this interaction, the student continued to express his dislike for fractions and requested to move on. The sight of fractions shocked him and he believed he was incapable of continuing the procedure.

Algorithmic replication with no confidence. After subtracting 4 from both sides, five of the six students said they "have to flip", "have to swap", or "have to turn it into multiplication and flip the fraction" when dividing both sides by 2/5 in order to isolate *x* in the equation $\frac{2}{5}x = 6$. Such terminology is algorithmic. The students showed no reasoning process but took it as a necessary procedure to remove the fraction 2/5, and they did it with no confidence. For example, student 001 said:

> *"…when you divide you have to flip stuff, …but I don't know if that's right because this is implying multiplication (2/5) times x so I divided it…and because this is division, you have to turn it into multiplication and flip the fraction".*

This shows that the student was aware of the procedure but did not derive any meaning from their steps. Thus, the student couldn't justify the steps which led to self-doubt.

Student 006 used the method "Cross Multiplication" for the equation $\frac{2}{5}x = 6$ and expressed knowing nothing but replicating the procedure that the student just learned from a tutor in the Math Lab. The student expressed:

> "*I would go 5 times 6 so that would be 30/5. And then I would probably go 5 into 30 and that would equal 6 but I don't know if that's right…well ok I was doing this just barely in the math lab. Um and I was struggling with it, I flagged someone and she came up and she was like you cross multiply boom boom boom she is like that's the easiest way to do it. And so that's why I just barely did then, if I, if she wouldn't have shown me I would have probably been very confused by this. I don't even know if that's right?...the bottom portion ended being…you know what? It should have just been 2 right? Because I cross multiply, this was, this would have been 30, and then this would have been 2…"*

This indicates that the student was simply trying to recall the steps that were shown by the tutor. The student's admission elaborates that the tutor didn't emphasize meaning by explaining why the steps were being taken and so the student resorted to memorizing those steps.

To summarize, the students replicated procedures and showed no reasoning or understanding. In addition, among the five students who used the "flip/swap" method, only one student was confident in his/her work. The other four students explicitly expressed that they weren't sure whether the procedure they employed was correct.

The division of fractions. There were several difficulties students encountered when dividing 6 by 2/5 in order to isolate *x* in the equation $\frac{2}{5}x = 6$. First, student 002 multiplied both sides by 5/2 rather than dividing both sides by 2/5. The student said, *"It made it easier to do it that way*" – which indicates that the student understood flipping as a method of simplifying the division of fractions. The student continued by expressing uncertainty whether to divide or multiply both sides so as to isolate x. The student said, "*I guess you could get the same answer dividing it…I am not sure if to get x by itself on this side if I would divide it over here or times it, so I am not sure about that.*"

Second, student 003 was not sure which number to flip when dividing 6 by 2/5. The student commented, "*I wasn't sure exactly if I needed 6 to remain the same or the 6/1 remain the same or <u>if I should have flipped that one instead of the other one</u>.*" This is another admission where the student shows concern over the division of fractions. The student had memorized the steps and failed to recall which fraction should have been flipped.

To summarize, the students favored multiplication more than division of fractions. The procedure of flipping a fraction when turning division of fractions to multiplication of fractions made four of the students worried, uneasy, and afraid of making mistakes.

Problem #10: Symbolic, Operational, and Strategic Difficulties

Students were asked to solve the equation $\frac{3x}{7}+\frac{1}{6}=\frac{1}{3}$ (problem #10). Only one student was able to solve problem #10 successfully. The other five students failed to solve the problem due to either their inability to begin the problem, operations overriding strategy, the position of the variable x, or the confusion of multiple ways to solve the equation. The students who encountered the four difficulties are shown in Table 5 below.

Table 5

Students and difficulties from problem #10

Difficulty/Concern	Frequency	Students
Inability to begin the problem	1	001
Operations overriding strategy	2	002, 005
Position of the variable x	3	002, 003, 006
Confusion of multiple ways	1	004

Inability to begin the problem. Student 001 did not try to solve the problem. The student said, "*<u>No clue</u>! I haven't done a problem like this in a long time…I would <u>I don't know where to start</u>.*" This shows that even when the strategy of isolating the variable is still applicable, the appearance of multiple fractions obscures the student from recognizing the strategy.

Operations overriding strategy. Two students added the two fractions (i.e., $3x/7$ and 1/6) on the left side of the equation even though they were not like terms since one of them contained the variable x. The strategy "isolating the variable" was overridden by the operation of adding fractions. Student 002 solved the problem successfully on the first attempt, but soon overthrew it, did it again, and used "the order of operations" over "isolating the variable." The student said:

> "*I can redo that one and cause I think I think that one is off, I definitely think I did that one wrong…let's do the order of operations, then go left to right…I am…would have to find the common denominator to add them first…*".

Once again, this demonstrates that having multiple fractions disrupted the strategy, causing the student to prioritize the addition operation of $3x/7$ and 1/6. Failing to group or combine variable terms also led the student to an incorrect answer of $25x/42$.

Student 005 was stuck and could not go further after adding the two fractions. The student stated, "*I think I am stuck...I don't know how to resolve the variable. I don't I don't know how to do this so…I just I don't remember."* The confusion of prioritizing the addition operation over the strategy of isolating the variable led the student to an unfamiliar place, which then caused failure to proceed with the problem.

To sum up, the attention to the addition of the two fractions ($3x/7$ and 1/6) distracted the two students from implementing the strategy "isolating the variable" (i.e., subtract 1/6 from both sides).

The position of the variable *x*. Student 006 successfully performed the first step of subtracting 1/6 from both sides and got $\frac{3x}{7}=\frac{1}{6}$. However, for the following step the student subtracted 3/7 from both sides rather than dividing both sides by 3/7. The student stated, "*I'm trying to isolate the variable and so I would minus 3/7 and go minus 3/7…*" This student seemed to wrongly conceptualize the term $3x/7$ as $3/7 + x$.

Student 002 and 003 were bothered by "$3x$" and believed the "$3x$" was a term as a whole and could not be separated. Student 003 explained:

"so now we are looking 3x over 7 equals 1/6, and, I need to divide 3/7 by 1/6 so then again um I am going to times 1/6 by 7/3, um, that would give me 7, that would give me 7/18, and I am not confident in that answer only because the x is with the 3".

The placement of x in the numerator caused the students to lose confidence in their answer. Particularly, the students' confusion in either treating *3x* as one term that must be kept intact or as a term that can be separated into two factors. Student 002 attempted the problem again but this time s/he added the two fractions on the left side of the equation and got $25x/42$.

In summary, having the variable x on the numerator of a fraction caused various difficulties for these three students. Some took $3x/7$ as $3/7 + x$, and some took $3x$ in $3x/7$ as inseparable. None of them saw $3x/7$ as $(3/7)x$.

Confusion of multiple ways. Student 004 stated, "*I would subtract 1/6 from both sides…1 minus 1 is 0 and then 3 – 6 would give me a –3*". This revealed that the student was unable to subtract fractions even though the initial step of "isolating the variable" was correct. The student was then confused about multiple ways of isolating the variable x:

> "*to get x by itself that's being divided so you are gonna times by 7 and those cancelled out…mmhh…so you times by the sevens and you have that 3x by itself and then you times the other side by the -3 times 7 you get -21 (ok) and then you divide by 3, divide by 3 and you get x = 7…<u>I am doing the dividing or I am supposed to be dividing or timesing…I know I am supposed to be doing one or the other but I am not sure</u>…*"

This illustrates that the student was confused by the multiple ways for solving the problem. For example, dividing both sides by 3/7, or multiplying both sides by 7/3, or as the student did - multiplying both sides by 7 and then dividing both sides by 3.

Discussion

The discussion presents the difficulties students experienced followed by instructional recommendations that could alleviate such difficulties. Specifically, the difficulty of removing the fraction from the variable, the difficulty of separating operations from strategies, and the difficulties resulting from the position of the variable x will be discussed.

The Difficulty of Removing the Fraction from the Variable

The first difficulty students encountered was to remove a fraction from the variable so as to isolate the variable. Part of the problem is that the students struggled with division of fractions. We found that all students were not able to provide reasoning or explanation about why division turns into multiplication when dividing fractions. Without any rationale, they used algorithmic terms such as "have to flip" or "have to swap" to guide their problem-solving steps. In addition, students were not able to justify whether to remove the fraction from the variable by multiplication or division. Student 002 is an example of this situation when she was solving $(2/5)x = 6$. The student was not sure whether to multiply or to divide the (2/5). In fact, the student got two different results by doing the problem using the two ways. Student 003 was also confused about which one to flip, 6/1 or 2/5. Even the student that used "cross multiplication" to solve the equation $(2/5)x = 6$ attested that he did not know if the method was right or wrong. The division of fractions and the confusion of multiple ways (whether to divide or multiply the fractional coefficient) were problematic for the students. These algorithmic steps were conducted without confidence and sufficient understanding - which leads us to believe that the students were relying on replication of memorized procedures.

Instructional recommendation. The first recommendation is that the teacher has to provide enough information about the rationale for the procedure of turning the division of fractions into the multiplication of fractions by the reciprocal of the second fraction. It is not easy to explain why a number divided by a fraction can be turned into the number multiplied by the reciprocal of the fraction. However, it does not mean it is impossible to explain it or that students will not understand. To illustrate, a teacher could explain that anything divided by 1/5 can be understood as being measured by 1/5 unit. A one-meter stick measured by another one-meter stick is one, but it is 5 if the one-meter stick is measured by a 1/5-meter stick. Hence, any number divided by a fraction "1/a" is equal to multiplying the number by "a". If a number is divided by 4/5, then after multiplying the number by 5, the result must be grouped into 4s because the number is divided by 4/5 not 1/5. Hence, the result found by multiplying by 5 must then be divided by 4. The above rationale explains why a number divided by a fraction is the same as the number multiplied by the reciprocal of the fraction.

Another concern is that the students did not have sufficient practice talking about the reasoning process or may even have been told to ignore the reasoning process altogether in favor of the skill. Teachers may not emphasize the reasoning but reinforce the procedure of "flip the fraction" or "multiplied by the reciprocal of the fraction" due to the effectiveness and efficiency of performing operations. However, the consequence of such teaching could result in students lacking confidence, lacking the ability to justify the procedure, being confused over procedures such as multiplying or dividing, or flip this or that fraction.

The Difficulty of Separating Operations from Strategy

The second difficulty students encountered was the conflict between completing operations and using a problem-solving strategy. When solving problem #10, some students were distracted by the addition of the two fractions on the left side of the equation. They added the two fractions first instead of subtracting the fraction 1/6 from both sides to isolate the variable. The operation on fractions (adding fractions) overrode the problem-solving strategy of isolating the variable.

Instructional recommendation. The students' confusion about the two notions could be from their lack of a clear distinction between strategies and operations. This could lead students to introduce a strategy in terms of operations. For example, the expression "subtract both sides to isolate the variable" contains both operations (subtract both sides) and the strategy of "isolating the variable." Strategy and operations are interwoven in the expression, and students might take the operations as the only or necessary way for implementing the strategy. In fact, to isolate the variable, you may add some number to both sides or multiply both sides by some number in solving various types of equations. Teachers need to provide a clear distinction between strategies and operations. The two could be confused, and consequently, the strategy could lose its effect in guiding one to choose appropriate operations for solving equations.

Another possible cause for the difficulty is that the students might not have had sufficient practice in strategic planning before they engaged themselves in operations for solving equations. It is important when solving equations that students think of a strategy first, and then think about the operations that are necessary to execute the strategy. If we, as teachers, do not ask students to practice strategic planning, the significant teaching in equation solving would be only operations.

The Difficulties Resulting from the Position of the Variable *x*

Students encountered the difficulty of including the variable x in the numerator of the fraction when they were solving problem #10. One student thought of the term $3x/7$ as $3/7 + x$, and subtracted both sides of the equation by 3/7 to isolate the variable. Another student thought of $3x$ as an inseparable term and therefore $3x/7$ as an inseparable fraction. The student consequently added the two fractions $3x/7$ and 1/6 on the left side, and even added the variable and number together ($3x/7 + 1/6 = 18x/42 + 7/42 = 25x/42$) to get $25x/42$. The last student successfully solved problem #10, but stated, "I am not confident in that answer only because the *x* is with 3."

Instructional recommendation. This difficulty could be arising from the misconception of the relationship between the variable *x* and the fraction 3/7. The concept of keeping a whole term $3x/7$ together apparently overrode the operational relationship between *x* and 3/7. Consequently, one student took $3x$ as inseparable and thus had difficulty separating *x* from $3x/7$. In fact, these students could only interpret $3x/7$ as an object rather than an operation. In general, the fraction 3/5 is often called "three fifths" rather than "3 divided by 5." Therefore, teachers should emphasize the operational relationship among the components of a fraction. For example, a teacher could help students interpret the fraction $3x/7$ operationally as 3 times x, divided by 7.

Implications

The results of this study have three important implications for mathematics curriculum, teaching, and students' learning. The study found that students had some difficulties solving linear equations in one variable that included fractions and tried to explain their possible causes. These possible causes can be used to identify ways to improve instruction in the three areas of students' difficulties that arose.

First, mathematics curriculum may include the teaching goal of justifying the procedure for the division of fractions (i.e., if a number is divided by a fraction then that is equivalent to multiplying the number by the reciprocal of the fraction). Teachers may need to directly address this issue to avoid rote learning without reasoning. The common goal for learning to solve one-variable linear equations, no matter whether fractions are involved or not, is to solve for the variable. Being able to justify operations is usually not the goal of learning to solve equations, compared to being able to perform a procedure and find the final answer. Therefore, meaning is lost in the process. That is why we advocate that the transformation of a number divided by the fraction $1/a$ to the number being multiplied by its reciprocal a needs to be justified. The next goal may be to justify the transformation from a number divided by the fraction b/a to the number being multiplied by the reciprocal fraction a/b.

Second, teaching and distinguishing the role of strategy and procedure, as well as their relationship in equation solving, may help avoid the error of operations overriding strategies. Operations are visible, and can be manipulated and justified. However, strategies are more abstract, communicated through natural language and operated as ideas or guidelines that are not as concrete as symbolic reasoning or manipulation. To distinguish strategies from operations, more complicated equations may be posed to students because strategies function more significantly when problems or solution procedures are more sophisticated. For example, problem #10 in the study is a good example of what strategy can do and how operations could override strategy.

Third, introducing the decomposition of a composite fraction (e.g., $3x/7$ means 3 times x, divided by 7) seems necessary to help students distinguish the role and relationship between the variable and its associated numbers. A composite fraction means a fraction that is composed of more than one number/variable in the numerator or denominator, for example, $3x/7$ or $(2x+8)/2$. The operations between the components of a composite expression could be implicit. Noticing and being able to decompose a composite fraction or expression could reduce students' uncertainty and increase students' operational skills in learning to solve equations.

In conclusion, the involvement of fractions in linear equations increases difficulty in the solving process because the number of computations also increase. Moreover, fractions create new difficulties whose possible causes highlight where improvement needs to occur. We identified that teachers need to improve and justify explanations of how reciprocals are used to turn division of fractions into multiplication. Another area of improvement is to help students see the differences between operations and strategies. And finally, students need to be taught how to break down a composite fraction, as an object, into its operational parts. Without improvement in these areas, students will likely rely on rote learning when it comes to solving linear equations that contain fractions.

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A Useful Observation about the Unit Circle

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A standard practice in most trigonometry classes is to present the sines and cosines of the special angles 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$ on a diagram of the unit circle similar to the one in Figure 1. In the diagram, the unit circle is parameterized by the counterclockwise angle θ measured from the positive x-axis by the curve $(x, y) = (\cos\theta, \sin\theta)$. Only the first quadrant of this diagram is shown in Figure 1, since the remainder can be filled out by simple applications of the Reference Angle Theorem.

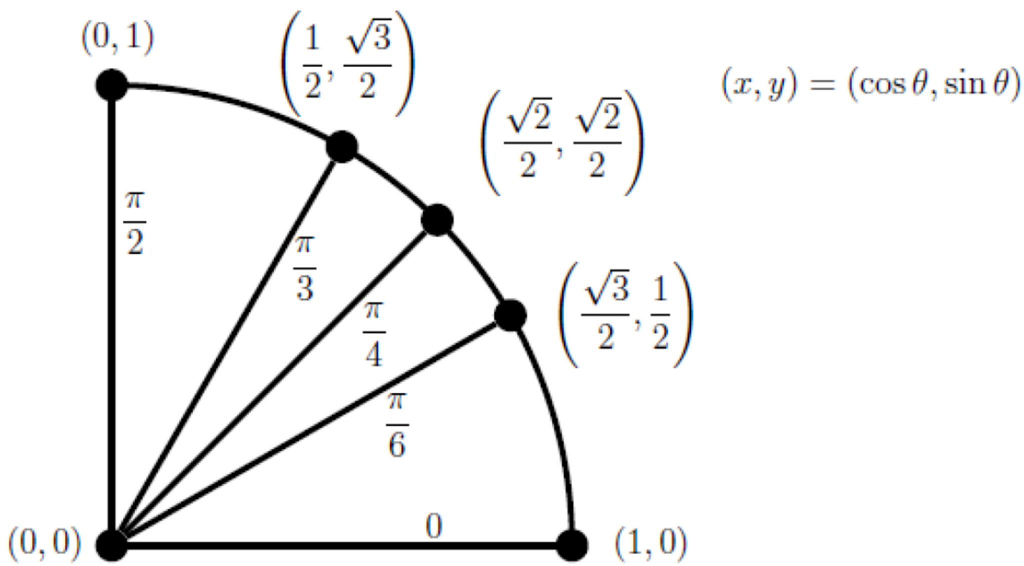

Figure 1: Displayed is a standard unit circle diagram of the first quadrant.

As a result, an obstacle which all trigonometry students face is to memorize the unit circle. While some students are able to memorize it without much difficulty, all too often students fail to memorize this sufficiently well as trigonometry students and struggle with it when it comes up again in Calculus or other more advanced courses. The most challenging part of the unit circle is knowing which angle, $\frac{\pi}{6}$ or $\frac{\pi}{3}$, gets the term $\frac{\sqrt{3}}{2}$ in the x-coordinate.

There are a lot of ways to remember these values, but one handy observation I made while helping my wife learn the unit circle when she took trigonometry over a decade ago quickly found its way into my classroom instruction of this subject as it made this memorization simple. It seems to be helping too. During the last time I taught this course, when I tested my students on the first quadrant of the unit circle during the final exam, over 80% of my students achieved a perfect score on that problem.

The observation is simple. Note that

$$\frac{\sqrt{0}}{2}=0, \qquad \frac{\sqrt{1}}{2}=\frac{1}{2}, \qquad \frac{\sqrt{4}}{2}=1.$$

Substituting these observations into the unit circle diagram produces Figure 2.

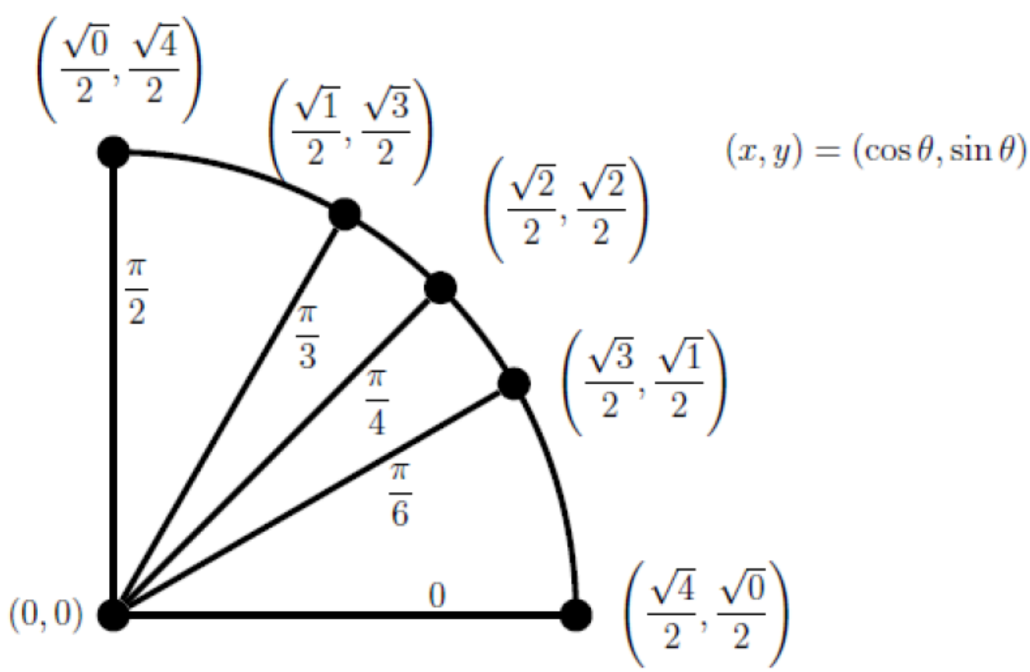

Figure 2: This is the same diagram as in Figure 1, but modified with the observation made in this article.

One can see that all the coordinates are of the same form, $\left(\frac{\sqrt{\bullet}}{2}, \frac{\sqrt{\bullet}}{2}\right)$. Moreover, as we rotate counterclockwise, what is in the square root in the x-coordinate counts down from 4 to 0 and that in the y-coordinate counts up from 0 to 4. This resolves many issues with memorizing these values as it links them together in a simple pattern. When I present it this way, I tell my students that if they can count to 4, they can memorize the unit circle.

This observation can be added to any discussion about the unit circle. However, I would caution that the use of this observation should not supplant a motivating discussion about why these values are the sines and cosines of these angles or why the unit circle can be parameterized this way. Otherwise, students may fail to gain the mathematical maturity those discussions induce. This observation should be treated as the coincidence it is and presented only as an aid to remembering these facts and not as a replacement for a theoretical discussion deriving them. But that said, the observation made here can be an immensely helpful way of memorizing the unit circle.

Graph Transformations by Variable Replacement

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A topic in algebra that students commonly find challenging to learn and perhaps some instructors find challenging to teach is the idea of graph transformations. This includes shifting, stretching, shrinking, and reflecting. The idea is basically to determine what the graph of a function like

$$y = af(bx + c) + d$$

looks like, given that we know what the graph of the parent function $y = f(x)$ looks like. It has been my experience in teaching this topic that students typically struggle with two main issues when trying to solve problems like this.

The first is why the horizontal changes always behave opposite algebraically to all of the vertical changes. For example, $y = f(x - 3)$ is a shift to the right (the positive horizontal direction) 3 units, while $y = f(x) - 3$ is a shift down (the negative vertical direction) 3 units. Even if the student can be convinced that this is what happens and that there are good geometric reasons for it, it is often still mysterious and difficult to remember.

The second is trying to figure out in what order to do all the different transformations if there is more than one involved. This often entails a little trial and error on the student's part. Usually there is more than one order that will work and plenty that do not. There are patterns that can be memorized, but again these can be difficult to remember.

In both cases, in a student's effort to memorize which way shifts and stretches go and which orders are right, they can often miss or forget a good reason why it must be the way they memorized.

Transformations of function $y = f(x)$ $(c > 0,\ k > 1)$	
$y = f(x) + c$	Vertical shift up c units
$y = f(x) - c$	Vertical shift down c units
$y = f(x - c)$	Horizontal shift right c units
$y = f(x + c)$	Horizontal shift left c units
$y = kf(x)$	Vertical stretch by k
$y = \frac{1}{k}f(x)$	Vertical shrink by k
$y = f(kx)$	Horizontal shrink by k
$y = f\left(\frac{1}{k}x\right)$	Horizontal stretch by k
$y = -f(x)$	Vertical reflection
$y = f(-x)$	Horizontal reflection

Table 1: Typical table of graph transformations.

Typically, textbooks and often instructors will present this material by first motivating the different parts individually. For example, one might explain that $y = f(x - 3)$ is a shift of the graph $y = f(x)$ to the right by 3 because the x values necessary to input into $y = f(x-3)$ to produce the same y-values as $y = f(x)$ are 3 to the right of those needed for $y = f(x)$. After going through explanations of each type of transformation, these methods then help students build a table to memorize similar to Table 1. However, there is a lot there to memorize and several things in the table are opposite for x and y which adds to the difficulty of memorizing such a table.

Many instructors combat some of this difficulty by creating very effective ways to motivate why the patterns for x and y are opposite. This is especially well done in (FF10) where the authors use overlays and discuss the differences between input transformations and output transformations so that students can see why x transformations seemingly behave opposite to y transformations. Another great method is found in (Kuk07), where the author discusses an inquiry-based method using reference graphs that has students discover the differences themselves. Ultimately both of these methods, as with many textbook descriptions, still result in a table like Table 1.

A really novel approach that still essentially results in the above table, but does not necessarily recommend students think of it that way is found in (Emb96). In this 20-year old paper, Embse uses the power of a TI-83 and parametric equations to show that the seemingly opposite patterns in x and y are actually one and the same if the function is viewed parametrically. For example, consider the function

$$y = (x-3)^2 - 7.$$

The parent function is $y = x^2$. To graph this will include a vertical shift down and a horizontal shift right of the parent function. As mentioned above, students may find it odd that the horizontal shift is in the positive horizontal direction and the vertical shift is in the negative vertical direction while both relevant signs are minus. However, if one thinks of this function as $y = t^2 - 7$ where $t = x - 3$ and then writes the function parametrically (which requires us to solve for x), we obtain

$$x = t + 3,$$
$$y = t^2 - 7.$$

In this way, one can now see why the x coordinate shifts right while the y coordinate shifts left. Embse describes how to do this for stretches, shrinks, reflections, and everything combined too. He recommends that students translate any function into a parametric version to identify all of the graph transformations (Emb96). And while it is true that once this is done, the correct transformations are apparent, the method itself is somewhat convoluted and requires students to be introduced to the idea of parameteric curves first. This may overcomplicate the issue and is probably why this method, while very clever, does not appear, at least from my experience, to have gained much traction in classroom use over the two decades since its publication.

In this paper, we present a method that can be attached to any of the above mentioned explorations or any other way to motivate Table 1. This method is one of variable replacement, which essentially utilizes the facts described by Embse in a way that does not require a discussion of parametric curves, but reduces the amount to memorize considerably. That is, the table I have my students memorize is much shorter than Table 1. Moreover, this method makes finding a correct order in which to perform multiple transformations practically foolproof as it uses algebra to check it. An added advantage to this method is that it lends itself to explaining why other algebraic changes to the equations of circles, ellipses, and other implicitly defined functions transform the way they do. We will present this by first describing the method, then solving an example, and finally discussing how it applies to implicit functions.

Replacing a Variable

Consider a parent function $y = f(x)$ and let $c > 0$. Suppose as well that we have motivated and convinced our students that the graph of the function $y = f(x) + c$ shifts the graph of f up (a move in the positive vertical direction) by c units and that the function $y = f(x - c)$ shifts the graph of f right (a move in the positive horizontal direction) by c units. The question then arises, or the instructor can bring it up, why the positive vertical shift is accompanied by a "$+c$", but the positive horizontal shift is accompanied by a "$-c$". The answer lies in the fact that the vertical variable y and the "$+c$" lie on opposite sides of the equation $y = f(x) + c$, while the horizontal variable x and the "$-c$" lie on the same side of $y = f(x - c)$. In fact, if we subtract c from both sides of $y = f(x) + c$, we arrive at the interesting observation that a positive vertcal shift actually looks like

$$y - c = f(x) \qquad \text{(positive vertical shift)}$$

while a positive horizontal shift looks like

$$y = f(x - c) \qquad \text{(postive horizontal shift).}$$

This observation is interesting because now the pattern is same for both variables. That is, we see that a positive shift by c in the direction of either x or y of $y = f(x)$ is obtained by replacing x or y in the equation by the expression $x - c$ or $y - c$, respectively. This is essentially what Embse was getting at with the idea of using parametric representations for the function (Emb96), but described in a way that does not require the parametric representation. A similar argument shows that negative shifts work the same way.

In fact, if we consider a positive shift by a negative number the same as a negative shift, then there is really nothing to gain by making the distinction between c being positive or negative. Instead, we can simply say a vertical shift by c means that we **add** c to all of the y-values. Similarly, a horizontal shift by c means that we **add** c to all of the x-values. Then the sign of c is irrelevant.

Putting this fact together with the observation above shows that given the graph of $y = f(x)$, replacing y with $y - c$ indicates a vertical shift by c or that we add c to all the

y-values. Similarly, replacing x with $x - c$ indicates a horizontal shift by c or that we add c to all the x-values. From this point of view, there is no difference at all in the effects and we have reduced all of the different translation information down to a single statement. Namely, that replacing a variable, x or y, in the equation $y = f(x)$ by that variable minus c indicates a shift in that variable's direction by c, or, equivalently, that we add c to all the values of that variable in the graph of $y = f(x)$ to obtain the new function's graph.

We get even more consolidation when we consider scalings and reflections. Now suppose that we have convinced our students that for $k > 1$, the graph of $y = kf(x)$ is a vertical stretch of the graph of $y = f(x)$ by k and that $y = f\left(\frac{1}{k}x\right)$ is a horizontal stretch by k. Again the same question arises why multiplication by k produces a vertical stretch, but multiplcation by $\frac{1}{k}$ produces a horizontal stretch. The answer is, in fact, exactly the same as for shifts. The multiplication for the vertical stretch occurs on the opposite side of the equation as the vertical variable, while the multiplication for the horizontal stretch occurs on the same side as the horizontal variable. By dividing by k in the vertical stretch equation we find that a vertical stretch actually looks like

$$\frac{1}{k}y = f(x) \qquad \text{(vertical stretch)}$$

while a horizontal stretch looks like

$$y = f\left(\frac{1}{k}x\right) \qquad \text{(horizontal stretch).}$$

Similar to shifts, we see again that the pattern is the same for both variables. That is, we see from the above that a stretch in the direction of either x or y in $y = f(x)$ comes from replacing x or y in the equation with $\frac{1}{k}x$ or $\frac{1}{k}y$, respectively. Again, this is essentially the parametric observation made by Embse (Emb96) but without the need of a parametric representation.

We also note that a shrink by a factor of k is really just a stretch by a factor of $\frac{1}{k}$ and in either case the stretch or shrink amounts to multiplying the variable in that direction by the stretching factor $\frac{1}{k}$, or, in other words, "scaling" by $\frac{1}{k}$. As such, instead of considering stretches and shrinks as different phenomona, we can just consider them both as scaling that coordinate. Reflections can also be thought of this way. A vertical reflection simply scales the y-values by -1 and a horizontal reflection scales the x-values by -1. Thus stretching, shrinking, and reflecting are really just scaling.

This fact together with the fact before shows that given the graph of $y = f(x)$, replacing y with $\frac{1}{k}y$ indicates that all of the y-values are to be multiplied or scaled by k in the new graph. Similarly, replacing x with $\frac{1}{k}x$ indicates that all of the x-values are to be multiplied or scaled by k in the new graph. And in either case, k can be any nonzero number, positive or negative. Once again from this point of view, all stretches, shrinks, and reflections are reduced down to a single statement. Namely that replacing a variable, x or y, in the equation $y = f(x)$ by that variable times $\frac{1}{k}$ indicates that we multiply by k all the values of that variable in the graph of $y = f(x)$ to obtain the new function's graph.

We collect the above observations into Table 2, which is the table I have my students memorize. Compared to Table 1, the simplicity and the reduced amount to remember with thinking about transformations as variable replacements is apparent. I often end up teaching this variable replacement method to students in college algebra, trigonometry, or caclulus because many, if not most, of them, even if they have been taught the content of Table 1 before, cannot remember much of it with any precision once they get to me. Those students

Transformations of $y = f(x)$ $(k \neq 0)$

Replacing variable	with	New Function	Graph Effect
y	$y - c$	$y - c = f(x)$ $y = f(x) + c$	vertical shift by c add c to all y-values
x	$x - c$	$y = f(x - c)$	horizontal shift by c add c to all x-values
y	$\frac{1}{k}y$	$\frac{1}{k}y = f(x)$ $y = kf(x)$	vertical scaling by factor of k multiply all y-values by k
x	$\frac{1}{k}x$	$y = f\left(\frac{1}{k}x\right)$	horizontal scaling by factor of k multiply all x-values by k

Table 2: Graph transformations by variable replacement.

are usually quite pleased with the reduction in what to remember. This is part of the reason why I think that this method could be of considerable use at the secondary school level.

However, the most impressive fact to me about thinking about the process this way, is that the algebra involved in replacing variables imposes a correct order in which to do the transformations. For example, suppose we know the graph of $y = f(x)$ and are asked to graph $y = f\left(\frac{1}{k}x - c\right)$ with $c, k \neq 0$. It is often confusing, especially when one is first learning about transformations, to figure out which happens first, the scaling by k or the shift by c. Experience or a memorized pattern can lead us to know that the shift should happen first and then the scaling. But even in that case, it can be difficult to remember that pattern especially considering in the graph of $y = kf(x) + c$, the scaling by k happens first and then the shift by c.

However, if one uses the variable replacement method from Table 2 and does one replacement at a time, the algebra forces a correct pattern. To see why, we will consider graphing

$y = f\left(\frac{1}{k}x - c\right)$ by comparing what the replacements would yield for $y = f(x)$ in both the incorrect and correct orders. Suppose we incorrectly scale first and then shift. This corresponds to the following replacements

$$
\begin{aligned}
y &= f(x) && \text{parent function} \\
y &= f\left(\frac{1}{k}x\right) && \text{replace } x \text{ with } \frac{1}{k}x\text{; scale } x \text{ by factor } k \\
y &= f\left(\frac{1}{k}(x-c)\right) && \text{replace } x \text{ with } x-c\text{; shift horizontally by } c \\
&= f\left(\frac{1}{k}x - \frac{c}{k}\right). && \text{distribute}
\end{aligned}
$$

From here, it is quite clear that this order is incorrect as it will graph $f\left(\frac{1}{k}x - \frac{c}{k}\right)$ which is not $f\left(\frac{1}{k}x - c\right)$. On the other hand, if we correctly shift first and then scale, we have

$$
\begin{aligned}
y &= f(x) && \text{parent function} \\
y &= f(x-c) && \text{replace } x \text{ with } x-c\text{; shift horizontally by } c \\
y &= f\left(\frac{1}{k}x - c\right) && \text{replace } x \text{ with } \frac{1}{k}x\text{; scale } x \text{ by factor } k
\end{aligned}
$$

and it is clear that we have arrived at a correct order since $y = f\left(\frac{1}{k}x - c\right)$ was exactly what we were looking to graph. In this way, a student can check by following through with the algebra (which only entails some mild simplification) whether a chosen order in which to do the transformations will actually acheive their desired goal.

Thus this method not only makes it easier to remember what each change does to the graph, but also provides a straightforward algebraic check to in what order the changes must be made. As long as we only do one replacement at a time, the algebra will indicate whether the order is correct. This essentially allows the algebra and not just a memorized pattern to help identify a correct way to transform the graph.

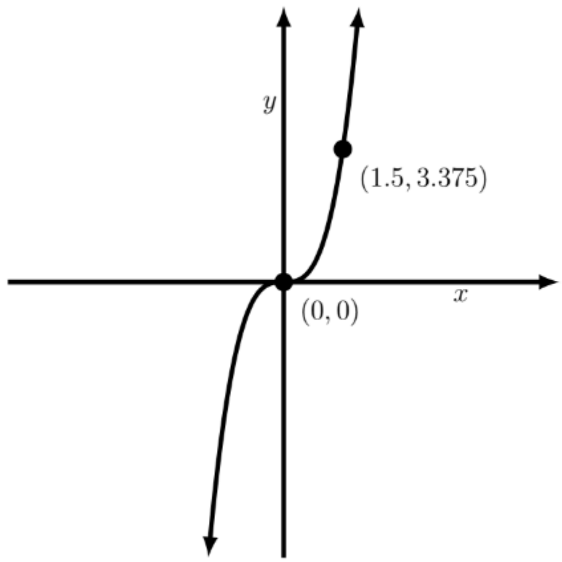

Figure 1: Graph of $y = x^3$.

Example

To illustrate the full power of this point of view, we will provide an example from start to finish. The function $y = x^3$ is graphed in Figure 1. In this example, we will graph

$$y = 2(-x + 5)^3 - 3.$$

To find the correct order of transformations, we start with the parent graph $y = x^3$ and make variable replacements step by step.

Function	Replacement	Graph Effect
$y = x^3$		parent graph
$y = (x+5)^3$	replace x with $x+5$	shift x-values by -5
$y = (-x+5)^3$	replace x with $-x$	scale x-values by $\frac{1}{-1} = -1$
$\left.\begin{aligned} \frac{1}{2}y &= (-x+5)^3 \\ y &= 2(-x+5)^3 \end{aligned}\right\}$	replace y with $\frac{1}{2}y$	scale y-values by 2
$\left.\begin{aligned} y+3 &= 2(-x+5)^3 \\ y &= 2(-x+5)^3 - 3 \end{aligned}\right\}$	replace y with $y+3$	shift y-values by -3

where the multiline braces indicate that the lines only differ by an algebra change, but are part of the same step. Notice that for the y replacements, we have to make the replacement that when moved to the other side of the equation will yield the function we desire. I think that this is the trikiest part of using this method. However, while it is not always obvious at first, after a few examples, my students have not seemed to find it particularly challenging to remember. Also note that on the last line, we arrived at the function we wanted to graph. If this does not happen, then an error has been made in either the order chosen or in the arithmetic or algebra used in some step. Thus students can check their work before they even begin graphing.

Now note that the changes to the graph follow the same order as the algebraic replacements. Thus the graph we want takes the graph of $y = x^3$ and (1) adds -5 to all the x-values, then (2) multiplies all the x-values by -1, then (3) multiplies all the y-values by 2, and finally (4) adds -3 to all the y-values. Two points are marked on the graph of $y = x^3$, $(0, 0)$ and $(1.5, 3.375)$, in Figure 1. Under the above transformations, these points move as

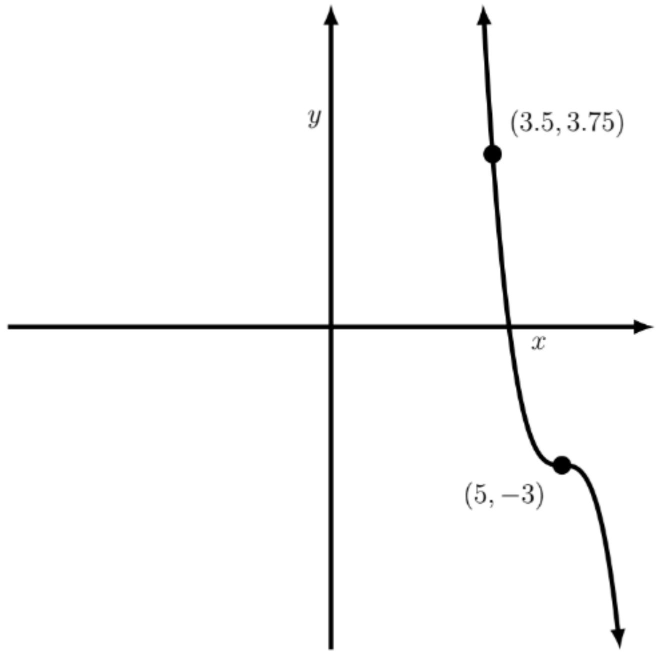

Figure 2: Graph of $y = 2(-x+5)^3 - 3$.

follows (each successive arrow represents the transformations listed 1-4 above).

$$(0,0) \to (-5,0) \to (5,0) \to (5,0) \to (5,-3).$$

$$(1.5, 3.375) \to (-3.5, 3.375) \to (3.5, 3.375) \to (3.5, 6.75) \to (3.5, 3.75).$$

The graph of the function $y = 2(-x+5)^3 - 3$ is in Figure 2 and we can see that the above points are in fact on the graph in the correct locations.

Implicit Functions

In this last section, we consider other advantages of viewing graph transformations as variable replacements. Already, it reduces the number of rules to remember from the typical 10 in Table 1 to 4 in Table 2 and it algebraically imposes a correct order to multiple transformations. However, unlike the description of transformations in Table 1, viewing transformations as in Table 2 also extends to implicitily defined functions.

For example, consider the equation for the circle centered at the origin of radius r,

$$x^2 + y^2 = r^2.$$

To shift the center from $(0, 0)$ to $(0, k)$ would simply shift all of the y-values by k. This graphical change corresponds to the algebraic replacement of y with $y - k$ and yields

$$x^2 + (y - k)^2 = r^2.$$

If we then want to the move the center to (h, k), we need only shift the x-values of the graph of the above function by h. This change corresponds to the algebraic replacement of x with $x - h$ and yields

$$(x - h)^2 + (y - k)^2 = r^2,$$

which is the familiar equation of a circle of radius r centered at (h, k).

As another example, consider the unit circle centered at the origin,

$$x^2 + y^2 = 1.$$

Now suppose we want the equation of an ellipse centered at the origin with vertical radius b and horizontal radius a. To obtain the graph of such an ellipse, we would take the unit circle and scale the y-values by a factor of b and the x-values by a factor of a. Scaling the y-values by a factor of b corresponds to replacing y with $\frac{1}{b}y = \frac{y}{b}$. Scaling the x-values by a factor of a corresponds to replacing x with $\frac{1}{a}x = \frac{x}{a}$. Making these two changes yields

$$1 = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2},$$

which is the well-known formula of an ellipse centered at the origin with horizontal radius a and vertical radius b. Next if we shift the center from $(0,0)$ to (h,k), then in the above equation, we replace x with $x-h$ and y with $y-k$ and obtain

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1,$$

which is the well-known equation of an ellipse centered at (h,k) with horizontal radius a and vertical radius b.

Thus viewing graph transformations in terms of variable replacements extends the concept of transformations beyond just explicit functions and leads to a more fundamental understanding of why all graphs transform the way they do under certain changes. By teaching students graph transformations via this method from the beginning, when they get to these later equations that are also important for students to remember, they now already have a method that tells them what the answer should be. This helps the student avoid memorizing another seemingly disconnected formula.

Conclusion

This method of graph transformation by variable replacement utilizes ideas that, similar to the parametric representations in (Emb96), show that the horizontal and vertical transformations do not behave oppositely and as such consolidates the "rules" to be memorized. However, the consolidation is done in a way that does not require added machinery, such as how parametric curves are needed to utilize the method in (Emb96), and actually informs later topics like implicitly defined functions.

Thus the major advantages of viewing graph transformations as algebraic variable replacements are really threefold: (1) there are fewer rules to remember, (2) a correct order is forced by the algebra, and (3) this approach extends to implicit functions as well. It is my experience that students usually find all three of these advantages quite useful and, after learning this method, are much more proficient at being able to transform a graph.

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Mindset

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Mindy Robison- Canyons School District

How does the use of assessment in my classroom foster a growth mindset? This paper will focus on specific strategies teachers can use to increase student confidence and willingness to make mistakes in order to learn mathematics. Specifically, we will focus on student feedback and the role of self-evaluation in assessment.

According to Carol Dweck, "The growth mindset is based on the belief that your basic qualities are things you can cultivate through your efforts" (p.7). Students and teachers with a growth mindset are concerned with improving, where those with a fixed mindset believe that intellect is "carved in stone" and are concerned about how they will be judged (Dweck, p. 6). Using the work of Carol Dweck, Jo Boaler applies the growth mindset to assessment in mathematics and identifies assessment for learning as a "form of assessment that gives useful information to teachers, parents, and others, but it also empowers students to take charge of their own learning." (p.94). Boaler identifies critical components of assessment for learning:

- Students need clear communication to know what they are learning and how they will get there through feedback
- Students need to be aware of where the are in the learning process

We will look at each component of assessment for learning and identify simple strategies teachers can utilize to create a classroom focused on using assessment to communicate using feedback and create awareness for student and teacher success.

Assessments to Communicate

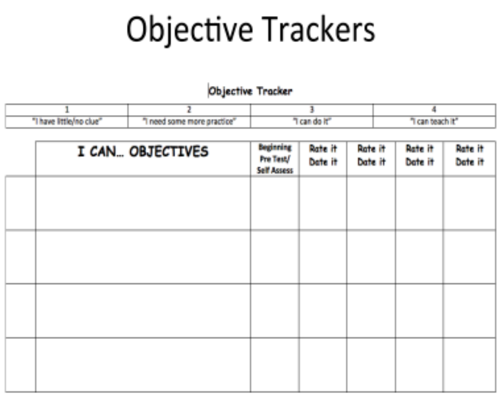
In the book *Principles to Actions*, the National Council of Teachers of Mathematics (NCTM) identified productive beliefs for assessment. The first two beliefs, "The primary purpose of assessment is to inform and improve the teaching and learning of mathematics," and "Assessment is an ongoing process that is embedded in instruction to support student learning and make adjustments to instruction (p. 91)" support Boaler's critical components for assessment for learning. Consider the following situation: A teacher gives a quiz to a student, after the student takes the quiz the teacher writes a C on the quiz and gives it back to the student, the student sees the C and throws the quiz away. Has the test provided a means to communicate to the student and the teacher what the student has learned and where he/she is going? Maybe, but is there a more effective way? Boaler, citing research done by Ruth Butler, found that students who received comments instead of a grade increased their performance significantly (p. 99). Students need to receive specific feedback on their assessments for learning that makes them more aware of where they are on their learning path. Students need to be aware of each mistake and provided feedback to learn from their mistakes. Boaler, quoting Dylan Wiliam, states, "Feedback to learners should focus on what they need to do to improve, rather than on how well they have done, and should avoid comparison to others" (p. 100). A student with a growth mindset seeks to get better through feedback, allowing for improvement. When a student and teacher in the same classroom have a growth mindset the potential for growth is exponential. *The simple strategy: give specific feedback on assessments for learning instead of a letter grade*.

Assessments to Create Awareness

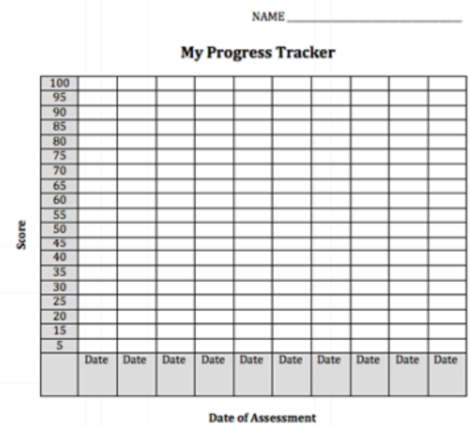
Teachers are not the only ones that can provide feedback and assess student understanding. Students can self-assess and evaluate their own performance. In John Hattie's work he identifies the zone of desired effect as effect size above 0.40, through his research he found that self-reported grades (students estimating their own performance) have the impressive effect size of 1.44 (p. 44).

In *Principles to Actions*, another productive belief on assessment is, "Assessment is a process that should help students become better judges of their own work, assist them in recognizing high-quality work when they produce it, and support them in using evidence to advance their own learning" (p. 92). In regard to assessment, Boaler explains, "In studies of self-assessment in action, researchers have found that students are incredibly perceptive about their own learning, and they do not over or underestimate it. They carefully consider goals and decide where they are and what they do and do not understand" (p.96). If students are able to accurately self-assess what tools do teachers need to provide to allow students this type of opportunity? One strategy, identified by Boaler, is "traffic lighting" where students are asked to put a red, yellow, or green cup on their work to assess their level of understanding of new work (p.98). This simple strategy allows students to take a moment and assess their understanding, as well as provides the teacher an opportunity to receive feedback and then adjust instruction accordingly.

Another strategy is to provide an "Objective Tracker" to students (see Figure 1) as used in Canyons School District. A teacher creates a list of objectives in the form of "I can" statements for students in mathematics using Utah State Core Standards. Students self-assess at different intervals during the learning process and rate themselves on a 1 to 4 scale on the "I can" statement.



(Figure 1)



(Figure 2)

Another self-assessment strategy that creates student awareness of their learning is having students graph their progress. For example, if students are taking benchmark tests three or four times throughout the year they will need to monitor their progress in the interim of the benchmarks. Students can use a simple graph to monitor their progress and monitor progress toward mastery (see Figure 2).

A final strategy suggested by Hattie is for students to create a goal called their "Personal Best" (p.165). Students set individual goals that are specific to their achievement. Setting "Personal Best" goals had high positive relationship to educational aspirations, enjoyment of school, participation in class, and persistence on task (Hattie, p. 165). Boaler, citing the work of Wilhelm & Black, explains that students need to move from passive to active learners taking responsibility for their own progress and teachers need to be willing to lose some of the control of what is happening. Boaler cites a teacher that says, "What it has done for me is made me focus less on myself and more on the children. I have had the confidence to empower the students to take it forward (p. 98). *The simple strategies: "traffic lighting," objective tracker, graphing progress, and "Personal Best."*

The growth mindset focuses on learning from mistakes and improving but we can only grow and learn when we know our mistakes. Boaler states that students are often unsuccessful "not because they lacked ability but because they had not really known what they were meant to be focusing on" (p. 97). If assessment is going to be a map to success both students and teachers have to clearly know the outcomes. Focusing on feedback, self-evaluation, and clear outcomes provides students the opportunity for assessment to become an integral part of improvement. Consequently, assessment no longer resembles the unproductive belief identified by NCTM as "something that is done to students" but a process to foster growth and understanding.

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Using Writing in the Mathematics Classroom

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Numerous studies have shown that incorporating writing into the learning process has a significant benefit in deepening understanding (Biancarosa & Snow, 2004; Graham & Perin, 2007; Baxter, Woodward & Olson, 2005). The majority of these studies have used written composition and described its effects on improving reading comprehension. Although there has been far less research conducted on the connection between writing and conceptual understanding in mathematics, it is likely that incorporating writing into mathematics instruction would have similar benefits by deepening student conceptual understanding of mathematical concepts. When students demonstrate conceptual understanding they are more able to use this knowledge to solve problems, use it flexibly, and avoid common misconceptions.

Additionally, it is clear that the Standards for Mathematical Practice call for students to engage in reading, writing, and speaking about mathematics. Mathematics instruction that aligns with the Common Core State Standards for Mathematics (CCSSM) (CCSSI, 2010) is demanding that students be able to communicate their thinking and ideas while building conceptual understanding of the concepts and ideas being learned. Students who articulate and justify their mathematical thinking and reason through their own and their peers' explanations will develop deep understanding that is essential to continued success in mathematics (National Council for Teachers of Mathematics, 2014).

This article describes two instructional strategies that can be employed to engage students in speaking and writing about mathematics as an avenue to deepening their understanding of various mathematics concepts. Although this particular lesson focuses on multi-digit subtraction with regrouping, the strategies presented can easily be modified to coordinate with any mathematics concept.

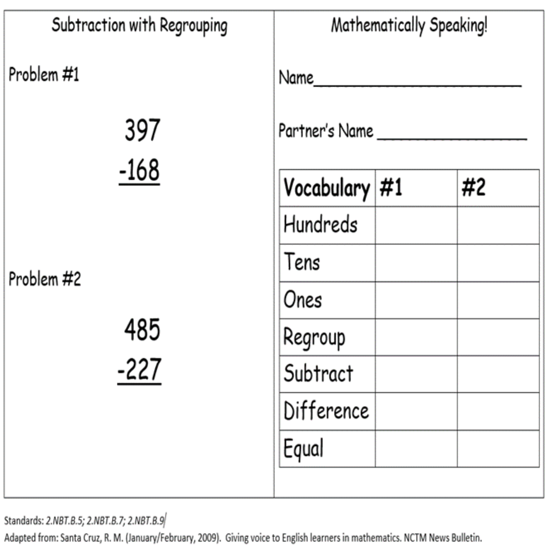
Background Information

A math lesson integrated with writing was presented during the second week of third grade as a review of students' understanding of multi-digit subtraction that required regrouping. We were interested in finding out which students understood the concept and which students would need additional instruction. We used two instructional strategies to investigate students' conceptual understanding: 1) Mathematically Speaking (Santa Cruz, 2009) and 2) a written response frame.

Strategy 1: Mathematically Speaking

The lesson opened with the teacher introducing the Mathematically Speaking template (see Figure A). First, the teacher modeled a multi-digit subtraction problem that required regrouping, similar to those found on the template. As the teacher explained and solved the problem the students were asked to keep track of which vocabulary terms the teacher incorporated into her verbal explanation. Upon finishing the modeled problem, the teacher then explained to the students that they would be working in partners to solve similar multi-digit subtraction problems. Partner 1 would solve problem #1 and partner 2 would solve problem #2. The students were asked to solve the problem and explain the process they used to do so. As part of their explanation, their partner would be tally marking which of the key vocabulary terms the students used during their verbal explanation. The partners were to encourage each other to use all of the vocabulary at least once during their explanation. Through requiring students to verbally explain their reasoning, students were able to "solidify and strengthen their understandings of mathematical processes and concepts because in the process of verbally explaining something to others, students often clarify for themselves what they mean" (Fogelberg et al., 2008, p. 57).

Figure A: Mathematically Speaking Template


Subtraction with Regrouping

Problem #1

397
-168

Problem #2

485
-227

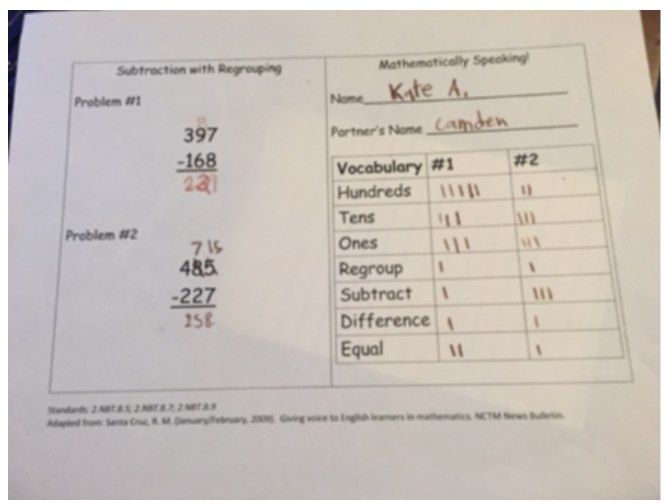
Mathematically Speaking!

Name____________________

Partner's Name ________________

Vocabulary	#1	#2
Hundreds		
Tens		
Ones		
Regroup		
Subtract		
Difference		
Equal		

Standards: 2.NBT.B.5; 2.NBT.B.7; 2.NBT.B.9
Adapted from: Santa Cruz, R. M. (January/February, 2009). Giving voice to English learners in mathematics. NCTM News Bulletin.



Strategy 2: Written Response Frame

After students completed explaining their thinking to their partner, students were then asked to write how they solved the problem. The teacher modeled how to complete the written response frame using the problem used at the beginning of class. Students were given two options for their written response: 1) open-ended response in which they were given a blank piece of paper to provide their response, or 2) the provided written response frame (see Figure B).

Figure B: Written Response Frame

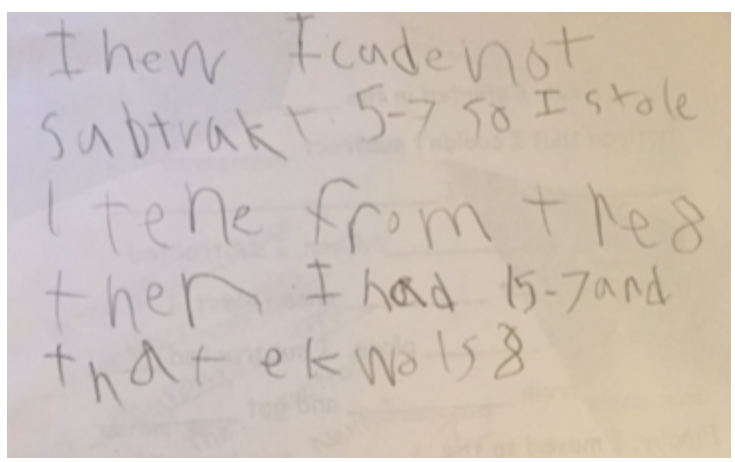
First, I started in the ________ place. I noticed that I couldn't subtract ________ (number + place value) from ________. So, I ________________________. Then, I subtracted the ones and got ________ ones. Next, I moved to the ________ place. I subtracted ________ (number + place value) from ________ (number + place value) and got ________ (number + place value). Finally, I moved to the ________ place. I subtracted ________ (number + place value) from ________ (number + place value) and got ________ (number + place value). The difference equaled ________.

The student responses were collected and used as a formative assessment to guide future instruction. Written responses provide greater insight into students' understanding, especially in comparison to purely numerical responses. Below are some examples of the students' responses.

Student Example 1: **Connor's Open-Ended Written Response**

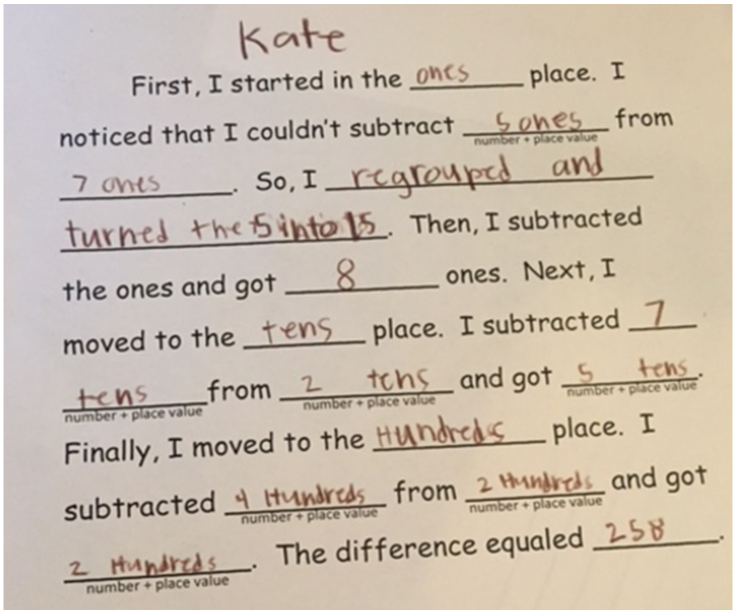
"I knew I could not subtract 5-7 so I stole 1 ten from the 8. Then I had 15-7 and that take away was 8.

His partial explanation demonstrates his understanding of regrouping.



Student Example 2: **Kate's Written Response Frame**

Note: Kate's explanation indicates that she took "7 tens from 2 tens" and "4 hundreds from 2 hundreds". This may be a simple error, but requires additional information to know for certain.

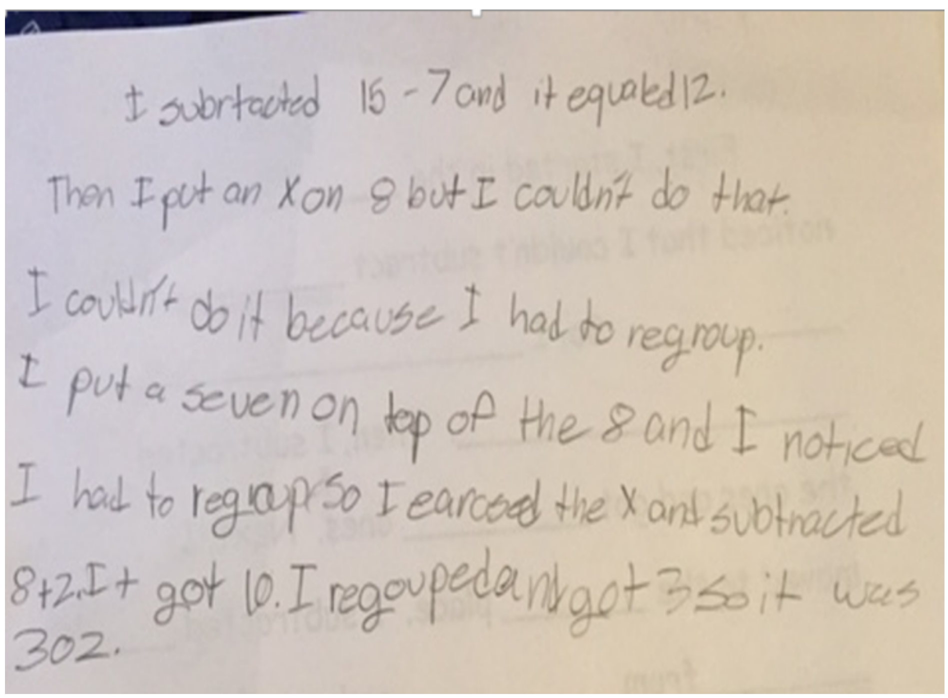


Kate

First, I started in the ones place. I noticed that I couldn't subtract 5 ones (number + place value) from 7 ones. So, I regrouped and turned the 5 into 15. Then, I subtracted the ones and got 8 ones. Next, I moved to the tens place. I subtracted 7 tens (number + place value) from 2 tens (number + place value) and got 5 tens (number + place value). Finally, I moved to the Hundreds place. I subtracted 4 Hundreds (number + place value) from 2 Hundreds (number + place value) and got 2 Hundreds (number + place value). The difference equaled 258.

Student Example 3:
Allison

Allison is a perfect example of how useful asking students to write about their thinking can be. The explanation provided insights into the student's misconceptions and lack of understanding.



I subtracted 15 - 7 and it equaled 12.
Then I put an X on 8 but I couldn't do that.
I couldn't do it because I had to regroup.
I put a seven on top of the 8 and I noticed
I had to regroup so I earced the X and subtracted
8+2. It got 10. I regoupedand got 3 so it was
302.

Reflecting on students' strategies

Through listening to students' verbal explanations and the collection of their written responses, we were able to gain useful formative assessment information. The student work samples were extremely useful in demonstrating concrete evidence of students' thinking processes and mathematical understanding, which in turn would be used to support group decisions and to adjust instruction in the areas of subtraction, place value, and regrouping.

Furthermore, as the students orally explained their reasoning to their partners they were able to clarify their thinking and justify their understanding.   We encourage you to use the Mathematically Speaking template and written response frames as avenues for facilitating students' mathematical reasoning. These instructional practices provide opportunities for students to demonstrate their conceptual understanding of mathematics concepts. Verbal and written demonstrations of mathematical understanding are invaluable for ascertaining students' thinking and determining next steps for instruction.

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Turn it Around: Culturally and Linguistically Responsive Teaching

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"Can you turn it around?" Nico, a second grade Latino English Learner (EL), was trying to solve the problem: "What plus three equals seven?" To help Nico, we asked him if he could use his turn-around facts (also known as fact families) to help him solve the problem. After asking if he could "turn [the problem] around," he took his piece of paper and slowly turned it around in a circle with his hand. Nico did not understand what we were asking. We realized that we needed to *turn around our own words and practices* to be more culturally and linguistically responsive to Nico's needs.

All students, regardless of ethnicity have their own unique cultural and language experiences. Their personal culture and language is shaped by their age, home life, traditions, school, and media exposure, such as videos or books. To address these concerns for equitable access to mathematics the National Council of Teachers of Mathematics (NCTM) suggests that students "should [not] receive identical instruction. Rather, [equity] demands that reasonable and appropriate accommodations be made and appropriately challenging content be included to promote access and attainment for all students" (Principles and Standards 2000, p. 12). As teachers, it is easy to assume that our students understand the language and context of mathematical word problems designed by curricula companies. Unfortunately, these assumptions can cause misunderstandings and frustration on the part of both students and teachers. Teachers can help minimize this by being more culturally and linguistically responsive. Although being culturally and linguistically responsive is an approach typically used to support ELs, the strategies in this article can be used to support *all* students in mathematics.

Linguistically Responsive

Students have a wide range of linguistic (language) abilities and understandings. Linguistically Responsive teachers cognitively consider the range of their students' linguistic abilities and actively advocate for teaching practices that support all students' understanding (Villegas and Freedson-Gonzalez 2008). Three strategies teachers can use when adapting mathematics tasks in response to children's linguistic needs are: (a) being aware of ambiguous mathematics vocabulary, (b) eliciting the use of student language, and (c) building up mathematics language through multiple representations.

Be Aware of Ambiguous Mathematics Vocabulary

Often, in mathematics, there are multiple terms for the same thing. For example, if I wanted my students to add two quantities, I might ask them to “add them,” “find the sum,” “combine the quantities,” “calculate the total,” or even “count all.” All of these phrases are synonyms for addition, which can be very confusing to an EL. If students are not familiar with a mathematics term used by the teacher, they may have difficulty understanding the task, even if they have the mathematical knowledge.

For example, we observed one student, Ana who struggled to solve a task which asked for the sum of two numbers. She asked the teacher is she should add, subtract, multiply, or divide. Once Ana learned “sum” is synonymous for “add,” she was able to quickly solve the problem. This was a simple misunderstanding with mathematics language that could have been prevented by helping Ana develop mathematical terms before the lesson.

Another issue with mathematics vocabulary is that the same words can be used to mean more than one thing. For example, consider the phrase *less than* in the following problem:

1. There were *less than* 3 black birds sitting on the fence.
2. There were 3 *less* blue birds *than* red birds.
3. The total number of birds is 3 *less than* 10.
4. Is the number of blue birds *less than* the number of black birds?

In this problem, *less than* is used to note a small amount (number 1), consider a smaller amount than another group (number 2), subtract (number 3), and make a comparison (number 4). Although students are not likely to see the same word used more than one way in the same problem, they may encounter multiple meanings across several problems which can cause misunderstandings and confusion.

Avoiding unfamiliar mathematical vocabulary is not linguistically responsive. A variety of mathematical terms are used in both state mandated assessments and content standards. Avoiding these terms can limit students’ personal growth in accessing mathematics. Teachers can help students’ future success by *building up* unfamiliar vocabulary using synonyms or child friendly language and then helping students use the specific terms during discussions and explanations. One idea is to pair vocabulary terms synonymously in call-response patterns to keep students engaged linguistically and reinforce mathematical concepts. For instance, if students hear the teacher call out “add” they answer “sum” or inversely, for “sum,” “add” is the answer. However, teachers should avoid pairing words that may have multiple meanings based on context (e.g., less-subtract).

Some ways to pinpoint words to attend to, is to preview the vocabulary terms used in students’ assignments and consider the terms most likely to be used during instruction. Being aware of ambiguous mathematics terms before a lesson is taught can help teachers anticipate and plan for potential issues in order to prevent confusion.

Elicit the use of Student Language

In order to be linguistically responsive, teachers need to also be aware of each individual student's vocabulary. This can be difficult as teachers usually have more than one student. The NCTM's Principles to Action calls for educators to elicit student language when posing purposeful questions as an excellent way for teachers to be linguistically response (NCTM 2014). Teachers can promote engagement by asking students to explain, in their own language, what the problem is asking and their solution methods (Caldwell, Kobett, and Karp 2014). As students explain their reasoning, the teacher can get a sense about which vocabulary terms they do or do not yet understand.

For example, Benjamin, a first grade EL, did not know the mathematical term *subtraction*; however, he used his own language to express his idea that "twenty *take away* ten equals ten." While discussing a task involving the remove of ten items from a set of twenty, Benjamin said "twenty '*no thank*s' ten equals ten." The *no thanks* term came from a familiar classroom phrase used to positively convey the message "don't do that." Benjamin obviously understood mathematical subtraction concepts beyond those he could articulate. By allowing him to explain his reasoning in his own words, the teacher gained an understanding of Benjamin's conceptual and linguistic abilities. After eliciting the use of student language, teachers can take advantage of these moments to help students make connections between specific mathematical terms and their own mathematics language (NCTM 2014).

Build up Mathematics Language through Multiple Representations

Language is a powerful tool. A single word, such as "ice-cream," can elicit a wide variety of home and school experiences. Teachers can use multiple representations such as visual models, gestures, intonations, physical models, and even music to help students make connections between their experiences and the language within mathematics problems.
For example, a louder voice with an accompanying gesture of open space between the arms to demonstrate "larger than ten" in contrast to a softer tone with a smaller gesture of space between the hands to demonstrate "less than ten." This activity can bridge language and overcome cultural barriers to reinforce the concept of *greater than* and *less than,* which is foundational for a fluid and flexible number sense (Gersten and Chard 2001).

Additionally, students are able to develop experiences and informal language by engaging with representations such as the pictorial number bond model in Figure 1 (a & b). These informal language experiences offer students multiple entry points into more formal mathematics language and instruction.

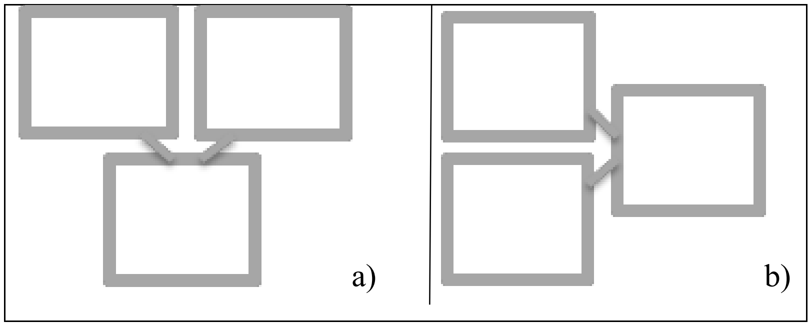

Figure 1. Number bond model in two orientations

To decrease confusion, it is important for teachers to describe the characteristics of a model like this to students. For instance, a teacher may say, “this model shows us three squares and two arrows.” This language with the model would draw the students’ attention to all three parts and how the arrows symbolize the joining of these parts (regardless of orientation) while building up student language around the operational concepts. Through frequent student discussion and use with this model, students eventually transition from informal language (*three squares and two arrows*) to mathematics language (*number bond* or *part-part-whole* representation). Once students begin using mathematics language they are able to develop more abstract and flexible addition and subtraction concepts with mathematics language.

Music can also help students access mathematics language for a variety of tasks. Familiar songs such as those that reinforce counting by five or ten can help students develop vocabulary while relying on “sing-song” patterns. Well known tunes can also inspire classroom adaptations for new contexts, languages, and cultures. For example, while working with the model in Figure 1, a first-grade class sang about the part-part-whole representations to the tune of “This Old Man.” The tune supported students as they developed formal mathematics language necessary to describe key elements for addition and subtraction.

Students love to create their own word problems around individualized cultural contexts. Students can use pictures, ten-frames, and other visual representations to clarify and illustrate their written word problems. These are both culturally relevant and enhance interest for students as they develop mathematics language. The use of multiple representations can help teachers “facilitate meaningful mathematical discourse” around the students’ word problems and solutions (NCTM 2014, p. 3). These discussions can help classes build a common cultural language for mathematics and other curriculum topics.

When students are encouraged to use multiple representations to explain and illustrate mathematics experiences they can develop a more comprehensive mathematics vocabulary and conceptual understanding of mathematics.

Culturally Responsive

Culturally responsive teachers help students make connections between their cultural experiences and mathematics contexts (Villejas 2002). Teachers need to explicitly draw from students' home and school cultures when helping them develop their mathematics understanding (Authors XXXX). Being aware of students' cultures can help teachers be more effective in this development.

Teachers should be aware that children from the same ethnicity might not share the same knowledge and experiences. For example, we adapted a start unknown algebra task ($\square+3=7$) for a first-grade EL to be more culturally responsive by using a piñata. Our rational was that the piñata reference might help our ELs more easily connect cultural knowledge to new learning (Caldwell et al. 2014). However, when one Latino student, Carlos, did not know what a piñata was and why there was candy inside of it, another student had to describe a piñata to Carlos. Carlos accessed the mathematics necessary to solve the piñata task, but did not have a cultural experience to draw from. As seen in this example, it is important that teachers do not always assume that just because students have similar cultures that they will have similar experiences. Thus, it is essential that previous experiences be understood by teachers to help students make these connections to new mathematics learning.

There are many different activities that teachers can use to make connections between students' previous knowledge and new learning, but it is essential these activities are relevant and interesting to students (Caldwell et al. 2014). One way teachers can make mathematics tasks relevant and interesting is through student surveys regarding home or school life and then graphing these results. As students count, graph, add, and subtract cultural contexts, they are offered a rich variety of connections between everyday language experiences and mathematics language experiences.

Another means in which to draw from students' own experiences to develop meaning is to use "you language." "You language" is a simple culturally responsive adaptation for word problems that can help students connect authentically with the meaning of the task. In *You Language,* the name of a person in a word problem is replaced with the word "you" (Artut 2015). For example, *you have 3 toys and lost 1* (rather than *Jeremy has 3 toys and lost 1*) places the student directly within the problem and connects them more directly with this experience. This student may envision their favorite 3 toys and then 1 toy going away, which offers an authentic and concrete understanding of an otherwise abstract situation. In this example, it should be noted that this adaptation did not change the level of difficultly nor the cognitive demand of the task.

Cooking can also be a highly engaging cultural and linguistic mathematics activity. For example, one group of first grade students bridged cultural and mathematics experiences while making homemade corn tortillas (see Figure 2). First, they counted the *bolitas de masa* (balls of dough); second, they predicted the geometric shapes the dough would be when flattened; and finally, they explored fractional concepts such as the division of regions and comparing halves to fourths when cutting up the cooked tortillas.

Figure 2. First-grade students making corn tortillas.

Teachers can be culturally responsive by incorporating students' personal experiences into their lessons. For example, when working with counters on relational equations, Elena was asked what the counters could represent. She chose cupcakes and explained how much she enjoys baking cupcakes. Adapting the task, by trading out an unfamiliar topic with one that was more familiar, helped Elena develop an interesting and relevant context for the mathematics problem.

Finally, it is important that educators continually try new ideas. Just because a situation is novel, does not mean that students won't enjoy it. Teachers can expand students' cultural experiences and build up students' vocabulary and schema by trying something new, watching a YouTube video, or looking at visual images. Once students make connections to the new experience, it can be used and understood in a variety of mathematics tasks.

Conclusion

Learning about your students – their interests and experiences – is critical when developing equitable mathematics tasks for *all* students. Educators can be more culturally and linguistically responsive by being aware of the students' language and culture, while also being aware of the standards' language and curriculum's contexts. Bridging these two facets of mathematics teaching is critical when teaching equitably. Teachers can effectively adapt mathematics tasks to be culturally and linguistically responsive by: a) building up cultural experiences and linguistic vocabulary; b) adapting tasks to be culturally relevant remove linguistic ambiguity; or c) replacing tasks where building up the context or language is not feasible. By building up, adapting, or replacing unfamiliar mathematical tasks, teachers can reduce misunderstandings and help students build a love for mathematics.

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Involving Immigrant Parents in the Mathematics Education of Their Children

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Parental involvement has been recognized as a contributing factor to students' academic success and to their progress in mathematics. Schools across the United States are welcoming a growing population of immigrant children, and their parents are often not engaged as expected by schools. To provide immigrant parents with opportunities to be involved in mathematics education, schools and teachers need to be knowledgeable of effective approaches to do so. This paper explores important findings and recommendations on immigrant parent involvement and suggests specific activities for mathematics teachers willing to initiate, maintain, and expand the participation of immigrant parents in the mathematics learning of their children.

Introduction

Parental involvement has been recognized as a contributor to students' educational progress and their greater success in schools (Fan & Chen, 2007; Olivos, 2004, Epstein, 2001). Parental support has been noted as a factor in students' progress specifically in mathematics (Sheldon & Epstein, 2005). At the same time, the lack of parental interest in and engagement with the school work of children has been suggested as a reason for immigrant students to lag behind in mathematics achievement compared to their peers (Civil, 2009). Critics maintain that immigrant do not provide sufficient support for mathematics learning at home and at school. In this paper, we first review the roots of this erroneous assumption about parental involvement of immigrant students. Then, an examination of the literature on immigrant parent involvement in mathematics education will provide the theoretical background for practical suggestions on what schools and teachers could do in order to involve effectively immigrant parents as contributors and participants in the mathematics education of children. Lastly, the paper offers a set of action steps and activities that teachers of mathematics across grade levels can use to invite and engage immigrant parents in the classroom. Throughout the paper, the reader will find the personal narrative of the author as an immigrant parent who has experienced firsthand the discussed issues.

Why focus on immigrants?

The popular definition of "immigrant" available in most dictionaries states is "a person who comes to a country where they were not born in order to settle there." (Wordnet, 2011). Arzubiaga, Nogueron, and Sullivan (2009) noted that this summative term includes voluntary immigrants, migrants, and refugees, and they suggest the term immigrant to emphasize the nature of resettlement (voluntary, work-related, or involuntary). In this paper, "immigrant" will be used summatively in reference to all individuals who were not born in the United States and relocated here with the intentions of permanent residence.

In the last decades, immigration has become the greatest demographic change factor in United States. The general trend of the resulting influx on the population throughout the nation is well documented and expected to continue. According to the U.S. Census Bureau (2008),

> The nation will be more racially and ethnically diverse, as well as much older, by midcentury, according to projections released today by the U.S. Census Bureau. Minorities, now roughly one-third of the U.S. population, are expected to become the majority in 2042, with the nation projected to be 54 percent minority in 2050. By 2023, minorities will comprise more than half of all children – and by 2050, the non-Hispanic Whites will be less than half of the total U.S. population. In 2050, the nation's population of children is expected to be 62 percent minority, up from 44 percent today. Thirty-nine percent are projected to be Hispanic (up from 22 percent in 2008), and 38 percent are projected to be single-race, non-Hispanic white (down from 56 percent in 2008).

These demographic changes are largely influenced by international immigration to the United States – and to Utah in particular. Current Census data shows that in the last ten years, the state of Utah welcomed about 530,000 new residents (U.S. Census Bureau, 2011). Of these, close to 157,000 reported Hispanic/Latino(a) as their ethnicity. In the last decade only, this greatly diverse ethnic group grew to 13% of the total Utah population (358,340 people). To compare, in 1990, the whole Hispanic population of Utah was less than 85,000. Previous Census predictions have Utah reaching a population of 210,000 people claiming Hispanic heritage in 2015; a sign that the change is happening significantly faster than ever expected.

57% of the last decade population increase in Utah for children who are under 18 years of age consisted of individuals considered a racial or ethnic minority. Currently, one of every four children in Utah schools is classified as a "minority." In public discourse, statements regarding low academic achievement and needs for special programs often accompany this label. Research and practice have been working on identifying actions that support the learning of the racially, culturally, and linguistically diverse students, and yet the educational response is not corresponding to the rate of demographic change. Our society and schools need to adequately address these changes by preparing the largely single race, monolingual teaching force to work effectively with children and parents from a variety of ethnic, cultural, and linguistic backgrounds. Administrators and teachers are in need to identify and adopt approaches that are in tune with the population changes, as the future of the nation will depend on the strong educational preparation of all of our students.

Parental Involvement in Schools

A spectrum of empirical studies shed light on aspects of parental involvement in schools that specifically affect student achievement. One obstacle to comparing their results has been the different ways to operationalize both parental involvement and student achievement (Fan & Chen, 2001). In studies that also considered racial or ethnic membership, the ways in which minority status has been labeled and represented have also been problematic. Claimed group membership according to official standard grouping is accompanied by a great within-group diversity in terms of nationality, ethnicity, languages, and traditions (including different educational traditions). While the broad racial and ethnic categories are needed for mostly statistical purposes, nowadays they fail to encompass the variety of characteristics within the larger groups while some are strongly associated with stereotypes and assumptions originating in the way the categories are constructed and used. This paper provides a summative review of the literature and the resulting set of strategies for initiating and maintaining for parental involvement that address within-group and between–group differences by looking for patterns that can provide a wide baseline on parental involvement trends that are common across the differences.

Parent Involvement in the U.S.

The first step in developing a comprehensive approach to involving immigrant parents in education is to establish what constitutes "parental involvement" in U.S. schools, as values and beliefs about the ways parents participate in education differ across countries and cultures. The expectations for parental involvement in schools across United States include a range of activities, from help with homework to volunteering at school activities. One of the most comprehensive frameworks (Epstein, 1995; Epstein, Coates, Salinas, Sanders & Simon, 1997; Sheldon & Epstein, 2005) organized these activities in six general groups. They include communicating with the school, volunteering at school, participating in school decision making, parenting, learning at home, and collaborating with the community. These types of involvement and their short descriptions are presented in Table 1.

Table One. Expected parent involvement in the U.S.

Type of parental activity	Description
Communicating	Designing and conducting effective forms of communication about school programs and children's progress
Volunteering	Recruiting and organizing help and support for school functions and activities
Learning at home	Providing information and ideas to families about how to help students with schoolwork and school-related activities
Parenting	Helping all families establish home environments that support children as students

Decision-making	Including parents in school decisions
Collaborating with the community	Identifying and integrating resources and services from the community to strengthen and support schools, students and their families.

Adapted from Epstein, 1995

Involvement of Immigrant Parents

Researchers have been exploring parental involvement as one potential contributor to the success of immigrant children in U.S. schools. Studies have found that racially and ethnically diverse parents value highly involvement in their children's education, and express strong interest in being involved in an active way (Hwang & Progestin's, 2010; Chavkin & Williams, 1993, as cited in Lopez & Donovan, 2009). At the same time, scholars report that parents of ethnically and linguistically diverse students are less likely to communicate regularly with their children schools, and are perceived by both teachers and administrators to be less involved and willing to be involved (Wong & Hughes, 2006). These parental behaviors are strongly attributed to lack of interest, lack of motivation, and lack of value associated with education (Lopez and Donovan). These conflicting findings force an exploration of the discrepancy between the professed willingness and interest for involvement of immigrant parents and the perceived absence such interest according to school faculty and officials.

Reports in the literature shed some light on the reasons behind such discrepancy. According to Chavkin and Williams (1993, as cited in Lopez & Donovan, 2009), immigrant parents may have expectations for parental involvement reflective of norms that are quite different from the ones established in the U.S. As a result, these parents expect different dynamics of the parent-school relationship since "what we learn through our culture becomes our reality, and to see beyond that is often difficult" (Chamberlain, 2005, p. 199). As an immigrant parent myself, I had to learn through experience the societal values and the corresponding expectations with respect to the role of parents in school in the U.S. Parents were expected to contribute in quite different ways in Bulgarian schools. For example, the established practice of having parents regularly present as volunteers in elementary classroom was completely foreign to me. I was not familiar with the unwritten rules on parent volunteering, and struggled to understand why it was acceptable to put children in the classroom in unequal position by having some parents present on a regular basis while other were understandably not able to be there during the school day. I gradually learned that this type of participation was determined by the history and nature of the educational system. This initial lack of knowledge and understanding of the educational tradition could be interpreted as unwillingness to be involved – and I made a conscious effort to find time, visit the classroom, and contribute similarly to the other parents. Still, I felt uneasy as I was not socialized in the form of parents-school relationship, and being present and participating in what has culturally been strictly teacher-dominated environment felt unnatural to me – a phenomenon recognized in the literature as related to cultural and ethnic identity (Civil, Planas, & Quintos, 2005). For similar reasons, I also hesitated to be the initiator of my involvement, a phenomenon well described by Wong and Hughes (2007): “although ethnic minority parents express a strong desire to be actively involved in their children's education, they are more likely than are majority parents to believe the school is responsible for initiating efforts and creating opportunities for parent involvement in school.” (p. 646). When expecting involvement of immigrant parents, educators need to recognize that not proactively seeking opportunities to volunteer in the classroom does not equal lack of interest, as parents may be coming from an educational system where the teacher is the one and only authority in the classroom. The lack of socially bound knowledge about the norms of parental participation while in the classroom, combined with a resulting sense of inadequacy not knowing what is expected from them, and the potential interference with the authority of the teacher should also be considered. In addition, immigrant parents may be sensitive about their level of English language fluency and their pronunciation and how it is going to be accepted by both teacher and students (Lopez and Donovan, 2009; Tinkler, 2002).

Other analyses of parental involvement of racially, ethnically, and linguistically diverse students confirm that the conflicts between what is valued by immigrant parents and is expected by U.S. schools and society are grounded in different cultural norms and beliefs about education. In an exploration of the mathematical practices of parents of African American students, Martin (2006) described the contradiction as rooted the commonly described images of these parents in education: “One of the limitations of the literature on African American parental practices and school involvement is that these parents are often portrayed as passive, lacking the kind of agency and advocacy that is accepted and expected as the norm for White, middle-class parents.” (p. 25). Similarly, in the review of parental practices of immigrants, Olivos (2004) observed that

> Conflicts between Latino parents and the public schools often lay in their differing views and values about education, particularly since these are the most "tangible" differences. Expanding this concept, I also believe that the tense relationship a bicultural (Latino, Asian, African, etc.) parent has with the U.S. public school system is negatively affected by the cultural biases and economic interests inherent within the institution of public education as demonstrated by its historic role of using its power to impose the values and wishes of the dominant culture onto bicultural student and parent populations. (p. 29).

Martin further explained that the pre-established norms often act as restrictions to appreciating the value of immigrant parental involvement: “The practices and behaviors that are idealized—for example, volunteering in schools and classrooms, helping students with homework, fund raising—are those against which all parents are judged.” (p. 25). Olivos added to this discussion by suggesting that Latino/a parents “lack the necessary appropriate avenues with which to access information concerning the education of their children and their rights as parents.” (p. 33) – and this lack results in misinterpreted expectations on both sides between schools and immigrant parents. The resulting gap and its interpretations by schools and parents may be the critical divide that needs to be address in order to effectively involve immigrants parents in education. Olivos (2004) summarized this conflict:

> The tense relationship a bicultural (Latino, Asian, African, etc.) parent has with the U.S. public school system is negatively affected by the cultural biases and economic interests

> inherent within the institution of public education as demonstrated by its historic role of using its power to impose the values and wishes of the dominant culture onto bicultural student and parent populations. (p. 29)

In sum, it has been established that parental involvement matters for the education of immigrant students – and it is also of importance to the parents themselves. If immigrant parents appear hesitant to get involved, it is for reasons different from lack of interest in or passion for education. A small sample of the contributing factors include a possible language barrier, including and especially in the specialized academic vocabulary; previously internalized cultural and educational traditions that are different from the ones in the dominant culture; and inability to attend school functions based on employment or other responsibilities. All of these factors are of importance when exploring ways to engage more immigrant parents in mathematics education and should be considered when building a comprehensive parental inclusion model.

Immigrant Parent Involvement in Mathematics

Research and practice strongly support immigrant parent involvement in the mathematics learning of their children (Lopez and Donovan, 2009; Sheldon and Epstein, 2005). Inclusion of immigrant parents not only positively influences the success of the individual student, but may have an effect on the educational system as a whole, as "parental involvement may be one means of reducing the achievement gap that exists between White students and some racial minority groups" (Jeynes, 2005, p. 263).

Strayhorn (2010) found that several forms of parental involvement affected positively student achievement in mathematics of Black high school children. Parental attendance of school gatherings and parent-teacher conferences, parent checks of mathematics homework, and visits of the mathematics classes were all related to higher student achievement. Strayhorn suggested that the information parents receive from attending school activities provided them with needed connections to what was needed for their children to succeed. In a study that examined the role of teacher-parent communication for student achievement in mathematics in a highly ethnically diverse school, Sirvani (2007) found that students who took home monitoring sheets of their mathematics work two times per week significantly outperformed their peers in the control group. The results were statistically significant on both testing and homework assignments, and lower performing students in the experimental group did significantly better than the lower performing students from the control group. These findings suggest that regular parent-teacher communication on mathematical content affects positively students' achievement, and that such practice may be one way to involve immigrant parents in the mathematics education of their children.

Lopez and Donovan (2009) also maintained that strong parent-school partnerships are in the heart of successful parental involvement, and "effective family–school mathematics partnerships consider the cultures in their community and develop appropriate mathematics content activities for parents as teachers." (p. 228). One role these partnerships play is to prove perspectives on mathematics education together with opportunities for parents to become familiar with the mathematical traditions, notations, and context of mathematics taught in the

U.S. This is a needed step as in terms of parental involvement in the mathematical learning of their children, since the ways in which parents have learned mathematics is one of the important factors that determine the ways in which immigrant parents engage in helping their children with mathematical tasks (De Abreu & Cline, 2005).

Civil and Bernier (2006) described a school-parent partnership where teachers and parents took the lead in teaching workshops on solving mathematical problems. Using a dialogical approach, the researchers engaged ethnically and linguistically diverse parents from a working class Latino neighborhood in leading mathematical discussions with teachers and other parents, where they were able to demonstrate their ways of knowing mathematics. During the project, parents and teachers engaged in heated conversations about the "right" ways of knowing mathematics as well as questioned each other's authority. Teachers insisted on the role of appropriate education training, while parents put more emphasis on exploring different ways of approaching mathematical content. Civil and Bernier concluded that while bringing parents as equal participants was an important and needed step to their involvement in their children mathematics education, there was also the need for further exploring how parents' intellectual contributions could be better utilized for the benefit of the teacher-parent collaboration. On a similar note, Menendez and Civil (2008) suggested informal parent workshops as one approach to involving parents more closely with mathematical content – and thus provide them with the tools and knowledge to engage in mathematical learning with their children. Important characteristics of the workshops were their voluntary attendance, flexible scheduling, and alignment of the workshops' themes with the content studies by the children of the attendees.

Ginsburg, Rashid, and English-Clarke (2008) found that engaging parents in the mathematics learning activities has the potential to also benefit their children. Adults who worked on improving their mathematical knowledge in order to help their children, began also learning from their children and as a result, were in a position to learn together with their children and thus to support them in a naturally occurring mathematical conversations. These reported advantages of the reciprocal parent-student relationship that occurred as a result of the focused parent learning should be considered by teachers when establishing activities to engage immigrant parents as participants in their children mathematical learning. I recall how surprised my children were to find out that the way I wrote and worked through mathematical problems was different from their teacher's and from their textbooks. I did not write the division sign the way they did in school, I did ask more questions than I gave "right" answers to the problems when they needed help, and occasionally mispronounced the mathematical vocabulary words. These were perhaps signs that "my" mathematics was not equally important and correct – and experience similar to what Civil (2009) described as "different forms of mathematics" (p. 1443) for immigrant parents. In working together, we discovered that "my" mathematics worked as well as the school one, and they were relieved when I began also writing and solving mathematical tasks in the way they were learning.

Sheldon and Epstein (2005) examined the effectiveness of 14 forms of parent-school mathematics partnerships, and found that all schools in their sample engaged in three of them: providing mathematics teachers' contact information to parents, individual parent-teacher conferences for struggling students, and regular submission of report cards to inform parents of students' progress. None of the schools in the sample organized activities for parents on Saturdays. Other activities included workshops for parents during daytime school hours or in the evening, awarding mathematics achievement certificates to students, inviting parents to school award ceremonies where mathematics achievement was recognized, making videotapes, game packets, and library materials on mathematics available to families, asking parents to volunteer as tutors, assigning homework to be completed with the participation of a family member, and developing and implementing assignments that incorporate real-life applications of mathematics in different occupations. The authors concluded that

> If schools hope to increase student test performance in mathematics, for example, they need to strategically plan family-involvement activities that encourage and enable interactions between students and family members relevant to the mathematics curriculum. Activities that engage many families and children in discussing and conducting mathematics at home are more likely than are other involvement activities to contribute to students increasing and maintaining their mathematics skills. (p. 204)

Lopez and Donovan (2009) suggested *Family Math Nights* as one form of creating parent-school connections that stimulated parental involvement in the mathematics learning of their children. They suggested using this type of community events as a culminating activity of a school-wide mathematics initiative, or as a separate venue for introducing parents to the mathematics curriculum. Lopez and Donovan promoted math nights as a community-building event that became the stimulus needed to stronger connect mathematical experiences at home and at school, and it resulted in positive attitudes of all involved. The authors further suggested that this type of partnerships provide a variety of involvement opportunities for parents, which could be a critical for involving immigrant parents. According to Lopez and Donovan, these nights "are leading the way in developing partnerships that respect language and culture while acquiring the language of mathematics and learning mathematics, effectively communicating to parents, and making school systems and resources accessible to parents and students." (p. 228)

Another contributing factor that affects immigrant parental involvement is the parents' existing level of anxiety about mathematics that could be further alleviated by the different ways in which mathematics is taught and explored in the U.S. compared to the native country of the parents. Anhal, Allexsaht-Snider, and Civil (2002) worked with three Latina parents who were able to observe the mathematics lessons in their children classrooms and then shared their reflections on the processes. Issues of different approaches to learning and teaching mathematics became apparent to parents, as well as their different levels of mathematical proficiency – and both should be carefully considered when initiating and planning immigrant parent involvement in the mathematical experiences of their children. A critical point of this study was the departure from a deficit view of immigrant parents by seeking their contributions as valuable resource on the educational process with their children's best interest in mind. Important points raised by the parents included the influence of previous mathematical learning experience in a different educational tradition on their perception of the teaching process, the role of mathematics teacher's enthusiasm about the content they were teaching for learner's engagement, and the influence of teachers' bilingualism on their ability to teach mathematics to non-native speakers. These observations signal parents' strong potential as contributors to the parent-school dialogue in improving the mathematics education of immigrant students.

Strategies for Immigrant Parent Involvement

Research suggests multiple forms of parental involvement in mathematics education that may be successful with immigrant parents. The strategies suggested below target the development of a solid parent-teacher relationship in general and the engagement of immigrant parents in mathematics learning in particular. These two areas are mutually supportive as they recognize the role of immigrant parents for the educational process and engage them as valuable contributors in a content area. This approach to parent involvement also reflects the approach suggested by Jeynes (2005). After performing a meta-analysis of the relation between parental involvement and student achievement, he found that

> Most notably parental expectations and style each demonstrated a strong relationship with scholastic outcomes. Thus, it was not particular actions such as attending school functions, establishing household rules, and checking student homework that yielded the statistically significant effect sizes. Rather, variables that reflected a general atmosphere of involvement produced the strongest results. Parental expectations and style may create an educationally oriented ambience, which establishes an understanding of a certain level of support and standards in the child's mind." (p. 262),

This view is further supported by Shah (2009) who claimed: "It is not the types of literature available to parents or the number of opportunities provided to be part of school activities that matter but, rather, how the social context makes parents feel about being involved." (p. 213). Shah extended the boundaries for parental involvement when she found that Latino parents "who have traditionally been seen as not interested in school activities, are more likely to be engaged in their child's school when they see themselves represented in governance and decision-making bodies." (p. 225). One way to translate these findings in the realm of mathematics education is to take a personalized approach that excludes any deficit assumptions and stereotypes and at the same time acknowledges and welcomes immigrant parent's previous experiences and knowledge. Initial steps toward achieving such atmosphere include the following:

1. Introduce immigrant parents to the forms of school-parent relationship established in your classroom and school, but also inquire about their experiences and expectations. As humans, our values are deeply and intrinsically embedded in our ways of making sense of the world, and their change is a process that involves learning and changing behaviors that are source of internal conflicts about remaining truthful to one's self. Demonstrating understanding and not devaluing the original parental beliefs on education instead of insisting on immediate and quick change is critical for one's willingness to embrace the new experiences.

2. Recognize and acknowledge that immigrant parents often hesitate to visit your classroom and get engaged in the ways you expect as a natural consequence of their previous ways of communicating with schools. Teachers' authority and autonomy characterize many school systems around the world, and immigrant parents may need time to understand the new

role they are expected to have in their new environment. To ensure transition and parent buy-in, consistently use activities that welcome parents' participation through indirect involvement – through mathematics homework or written communication that requires some form of parental response, and build inquiries about parents' previous mathematical experiences into the class work. This approach will supportive of the process of parents' gradual experiential learning about American education.

3. Change the tone and the content of conversations on immigrant parent participation. Is it really lack of parental involvement, or is it that we have not provided the opportunity and the knowledge for the immigrant parents to get involved? Do we expect them to know intrinsically the expectations and traditions with respect to parental engagement in education without presenting them with opportunities- and the time - to learn about them?

4. Choose communication forms that allow for best understanding. While communication with parents is important and valuable, contacting parents over the phone may not be the best option at first, due to a possible language barrier. Listening in the absence of body language and other visual cues is one of the most challenging language skills to develop. The inclusion of academic mathematical content in this mode of communication further complicates the challenges for parents. Unless there is an interpreter available to help, do not make a first contact over the phone if possible. A form of written communication would work better if a face-to-face meeting is not an option. Even with an available translator, using the phone to establish connection with parents immediately depersonalizes it. In addition, parents who have not experienced this type of teacher and school communication may consider your call a sign of academic trouble rather than an invitation to be involved in the day-to-day learning of mathematics.

5. Recognize that immigrant parents who are in the process of learning how to navigate the educational system do advance in their learning of language and culture – and with this change, your relationship will also grow and be open to new forms of parental engagement. The parent who needed a translator during your first meeting may not need one next time you meet. Similarly, by providing opportunities to become engaged with, share, and learn about the mathematical experiences of their children, you are putting down building blocks for other forms of participation.

6. One starting point to including immigrant parents in mathematics education that is often overlooked is engaging them in learning what content is taught and learned at different grade levels in elementary schools, and for middle and high school age children – understanding each course content, course sequencing, and tracks. Especially at times when changes are occurring – as with the starting implementation of the Common Core – parents need to be informed in order to be included. Again, they may not have previous experiences to build on – for example, if they as learners were exposed to a national curriculum - and schools need to reach out.

Activities for Engaging Immigrant Parents in Mathematics

The National Council of Teachers of Mathematics (NCTM) has recently set forth a number of recommendations for involving Latino/a parents in mathematics based school partnerships (NCTM, 2010). The suggested activities summarize most recent research and closely tie it with some of the traditionally strong values of Latino families. The NCTM document further supports the strategies just discussed in that considerations of culturally specific beliefs and expectations need to be an integral part of the efforts for successful immigrant parent involvement in mathematics education. They need to be taken in consideration if schools are seeking to engage immigrant parent in ways that are truly committed to the students' success.

The set of suggestions below are geared toward teachers who want to initiate activities where parents and students work together and use mathematics in context in order to provide opportunities for parental involvement. These activities as suggested as a way of building a bridge between parents' previous mathematical knowledge and ways of learning mathematics and their applications in relation to mathematics curriculum and pedagogy in American schools. As already suggested, this process takes time and mutual understanding to develop. A critical component here is the effort to sustain and expand the parent-teacher connections that are being built – as students' mathematical knowledge advances, the need to continue cooperation with parents in order to provide solid support at home becomes even more important.

Table Two includes a starting set of activities for engaging immigrant parents in the mathematical learning of their children in order to initiate and maintain involvement. The relationship sought here is a sustained, mutually beneficial communication that emphasizes parents' funds of knowledge and relates them to the mathematics curriculum. In addition, these teacher-initiated actions engage students in involving their parents are a resource for important mathematical knowledge recognized by the school and the teacher. They are appropriate for the elementary as well as the middle and secondary level.

Table Two: ***Activities***

General parental involvement steps	Mathematics-related activities

Approach parents with an invitation to take part in the learning process. Start with smaller, doable actions or activities that add to the daily mathematical activities – and gradually keep expanding their scope.	Have students write a (bi-weekly or monthly) note to their parents where they explain what they have learned in their math class, and how this knowledge is applicable. Ask for students to return the notes with a comment or signature from the parent. Implement a weekly "tracking" folder with a sheet listing the mathematics homework for every week or two. Request that parents sign to acknowledge homework completion. Then, include these sheets in students' portfolios and use at parent-teacher conferences when discussing students' progress. Send home regular updates on students' progress – and ask for some type of parental acknowledgement that they have seen them. Again, use them when you meet with parents to show that keeping parents informed and involved is part important for your class.
Inquire about and incorporate parents' funds of knowledge	Think "out-of-the textbook" and have students rewrite textbook problems in the context of their family – using their names and relationships, and to find out if these problems make sense to their parents – and in what way they would formulate a similar problem if they don't? Assign students to interview a parent on how they use or used mathematics in their daily activities and/or past and present occupations. Provide a short set of interview questions for students to use. The results on how and how much we use mathematics may be surprising for students and parents alike. You may find out stronger mathematical backgrounds than you expected for many immigrant parents. These stories will also provide you with information on possible connections with parents' mathematical experiences in the future. Ask students to include their parents when solving problems and to inquire about ones that exemplify mathematical concepts and originate in their home, traditions, or culture. What kinds of coins and bill denominations are used in the native country? What are some ways to work effectively with the metric system? How do you go from Fahrenheit to Celsius and vice versa? What games use mathematics? These parental contributions will have an illuminating effect on how we use Bishop's six universal mathematical behaviors (measure, design, count, locate, play, or explain) when using mathematics (Bishop, 1988).

Use activities that reinforce the development and use of mathematical language and overcome potential linguistic barriers.	Incorporate mathematical dictionaries as part of students' daily learning of mathematics – and ask for parent participation in the development (either through contribution or to sign completed ones on a regular basis). Teacher and students who speak a language different from English will be able to identify cognates or patterns in vocabulary, and the dictionaries may help with vocabulary recall and use and with the formation of mathematical connections across languages. Parents may also find similar connections for themselves. Include mathematical notation as part of the dictionary. Ask students to inquire about the ways their parents learned to denote operations, decimals, fractions, etc. and to include a sample problem in the dictionary.

Conclusion

Research and practice agree: when schools and parents work together, students' mathematical progress and achievement improve. However, the relationship between schools and immigrant parents is not always established and developed effectively, as there are a number of disconnects that prevent it from being as fruitful as possible. Some activities that could reform and improve the relationship have been identified and proposed in the literature, but there is a lack of specific teacher-led actions that teachers of mathematics can immediately use to involve immigrant parents. The activities offered in this article are geared toward building an ongoing relationship between mathematics teachers and immigrant parents. They can be immediately incorporated in the daily instruction and offer opportunities for further extensions in order to establish broader immigrant parent participation. Action research studies in classrooms where these strategies are applied could provide needed details on their effect on students' achievements as well as parents' response to their use.

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A Slightly Different Proof for the Limit of a Product

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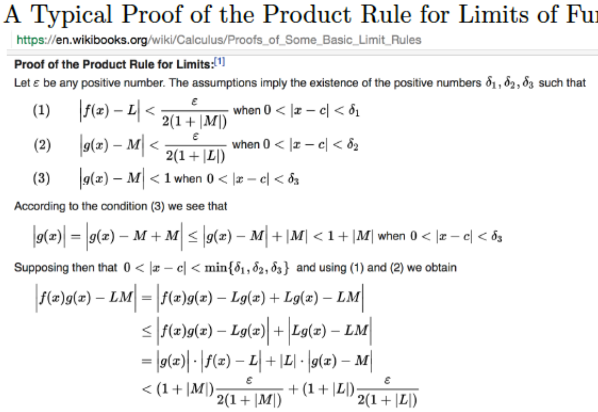
The theorems on the limit of a product of (of functions or sequences) say that 'the limit of the product is the product of the limits'. If one factor approaches A and the other approaches B, their product approaches AB. One fact is the basis for the result in both settings. We must be able to make a sufficiently close to A and b sufficiently close to B to guarantee that ab is within any prescribed positive tolerance $r > 0$ of AB. Here we seek to develop a more concrete, sharp, and deeper discovery based perspective on this key estimate upon which the proofs of those theorems are based.

Standard proofs of the sum and product rules for sequences and functions are often obscured by unnecessary and redundant inclusion of domain conditions that are not specific to the operation being analyzed, e.g., $|x - a| < \delta$ and $n > N$, that disguise the essential issues. For the sum, the operation specific question is "How close must x be to A and y be to B for $x + y$ to be within r of $A + B$?" (Answer: It is sufficient to keep x within $w_A r$ of A and y within $w_B r$ of B, where w_A, w_B are any two weights satisfying $0 < w_A$, $w_B < 1$, $w_A + w_B = 1$, or just $w_B = 1 - w_A$. Equal weights $w_A = w_B = \frac{1}{2}$ are conventional in theoretical proofs, but miss the opportunity to find α that gives optimally large domains in which $f(x) + g(x)$ or $a_n + b_n$ are within r of the $A + B$.

For the product, the operation specific question is, "How close must x be to A and y be to B for xy to be within r of AB. Together, these concrete estimates constitute the bulk of the technical work required to validate polynomial differentiation, and it seems a shame to deprive students of the opportunity to analyze and discover the answers for themselves. (Spoiler alert!: We will now describe a visual and inquiry based path to obtain the answer that is different in spirit from more standard presentations such as the two below.

Figure 1. A Typical Proof of the Product Rule for Limits of Functions [1].



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Proof of the Product Rule for Limits:[1]

Let ε be any positive number. The assumptions imply the existence of the positive numbers $\delta_1, \delta_2, \delta_3$ such that

(1) $\left|f(x) - L\right| < \dfrac{\varepsilon}{2(1 + |M|)}$ when $0 < |x - c| < \delta_1$

(2) $\left|g(x) - M\right| < \dfrac{\varepsilon}{2(1 + |L|)}$ when $0 < |x - c| < \delta_2$

(3) $\left|g(x) - M\right| < 1$ when $0 < |x - c| < \delta_3$

According to the condition (3) we see that

$\left|g(x)\right| = \left|g(x) - M + M\right| \leq \left|g(x) - M\right| + |M| < 1 + |M|$ when $0 < |x - c| < \delta_3$

Supposing then that $0 < |x - c| < \min\{\delta_1, \delta_2, \delta_3\}$ and using (1) and (2) we obtain

$$\begin{aligned}
\left|f(x)g(x) - LM\right| &= \left|f(x)g(x) - Lg(x) + Lg(x) - LM\right| \\
&\leq \left|f(x)g(x) - Lg(x)\right| + \left|Lg(x) - LM\right| \\
&= \left|g(x)\right| \cdot \left|f(x) - L\right| + |L| \cdot \left|g(x) - M\right| \\
&< (1 + |M|)\frac{\varepsilon}{2(1 + |M|)} + (1 + |L|)\frac{\varepsilon}{2(1 + |L|)} \\
&= \varepsilon
\end{aligned}$$

Figure 2. A Typical Proof of the Product Rule for Limits of Functions [1].

4. CONTINUITY DONE RIGHT 61

Using Lemma 4.6a) and taking $\epsilon = 1$, there exists $\delta_1 > 0$ such that $|x - c| < \delta_1$ implies $|f(x)| \le |f(c)| + 1$. There exists $\delta_2 > 0$ such that $|x - c| < \delta_2$ implies $|g(x) - g(c)| < \frac{\epsilon}{2(|f(c)|+1)}$. Finally, there exists $\delta_3 > 0$ such that $|x - c| < \delta_3$ implies $|f(x) - f(c)| < \frac{\epsilon}{2|g(c)|}$. (Here we are assuming that $g(c) \neq 0$. If $g(c) = 0$ then we simply don't have the second term in our expression and the argument is similar but easier.) Taking $\delta = \min \delta_1, \delta_2, \delta_3$, for $|x - c| < \delta$ then $|x - c|$ is less than δ_1, δ_2 and δ_3 so

$$|f(x)g(x) - f(c)g(c)| \le |f(x)||g(x) - g(c)| + |g(c)||f(x) - f(c)|$$
$$< (|f(c)| + 1) \cdot \frac{\epsilon}{2(|f(c)| + 1)} + |g(c)|\frac{\epsilon}{2|g(c)|} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

A former student, Jeter Hall took the intended route of isolating the question of how close must x be to A and y be to B to guarantee that xy is within r of AB. In a course, it is even better to begin more concretely by asking how close must x be to 2 and y be to 3 for xy to be within 0.01 of 6. If a student can discover a systematic answer that can be correctly adapted when the specific values become arbitrary, then they have solved the crux of early calculus for themselves, and are empowered to continue doing so in more advanced settings. Jeter took that aspect of mimicking the sum rule approach further than intended by setting them equal. He asked for an $R > 0$ for which

$$A - R < a < A + R \text{ and } B - R < b < B + R, \text{ implies } AB - r < ab < AB + r\ .$$

Now we also borrow a scaling approach to analyzing the limit of convolution means [3] to take Jeter's simplification a step further. What was shown in [3] is that the if a limit law can be proven for the special case of $A = B = 1$ and the operation is bilinear, e.g., multiplication, then the case of general A and B will easily follow from the special case.

For this reason we ask, given $r > 0$, for an $R > 0$ for which

$$1 - R < a < 1 + R \text{ and } 1 - R < b < 1 + R, \text{ implies } 1 - r < ab < 1 + r\ .$$

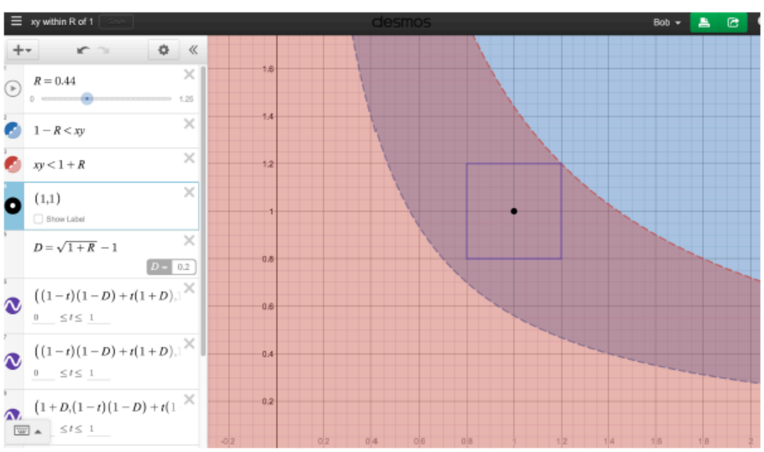

Figure 3. $1 - r < xy < 1 + r$.

Considering some concrete products of .8, , .9, 1.1, 1.2 we observe that those products farthest from 1 in each direction, $(.8)^2$ and $(1.2)^2$ occur when both factors are individually farthest from 1 in the same direction. If we impose restrictions that both a and b must be positive (distance from 1 less than 1) for any two such numbers, we can label them so that $a \leq b$, so then $a^2 \leq ab \leq b^2$.

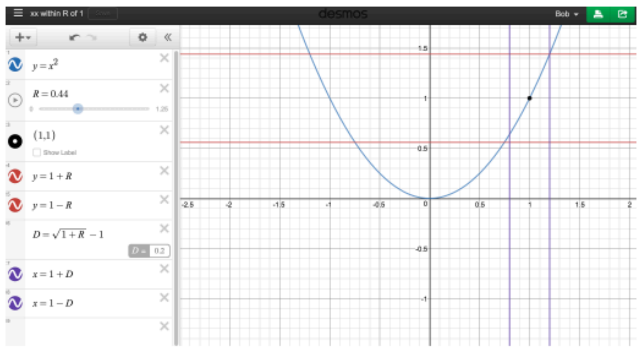

Figure 4. $1 - R < x^2 < 1 + R$.

We also see that $(1.1)^2$ and $(1.2)^2$ are farther from 1 than their counterparts to the left, $(0.9)^2$ and $(0.8)^2$. This suggests that if $(1 + x)^2 < 1 + r$ then $1 - r < (1 - x)^2$. To avoid circularity, we must be careful to not invoke any derivative based concavity properties to show this. If we succeed, then the bound $|x| < R$ we seek will be given by setting $(1 + R)^2 = 1 + r$, or

$$R = \sqrt{1 + r} - 1, \tag{1}$$

For technical convenience, to maintain positivity of all quantities greater than the lower bound $1 - r$, we can require that $r \leq 1/2$ because restricting the distance from ab to 1 to less than $1/2$ will satisfy any looser restriction. So as a first step, we will replace r by $\min\{r, 1/2\}$.

Using only the pre-calculus property that multiplying both sides of a strict or weak inequality by a positive number preserves that inequality, we can show that if both a and b are within a distance R to the right of 1, then ab will be within r to the right of 1:

$$\text{if } 1 \leq a \leq b < 1 + R \text{ then } b \leq ab < b(1 + R) < (1 + R)^2 = 1 + r. \tag{2}$$

To obtain the corresponding lower bound by the corresponding sequence of inequalities:

$$\text{if } 1 - R \leq a \leq b \leq 1 \text{ then } 1 - r < (1 - R)^2 < a(1 - R) < ab \leq a, \tag{3}$$

all but one again only require preservation of order by positive scalar multiplication. But one,

$$1 - r < (1 - R)^2 = (1 - (\sqrt{1 + r} - 1))^2 = (2 - \sqrt{1 + r})^2, \tag{4}$$

requires a bit of additional care.

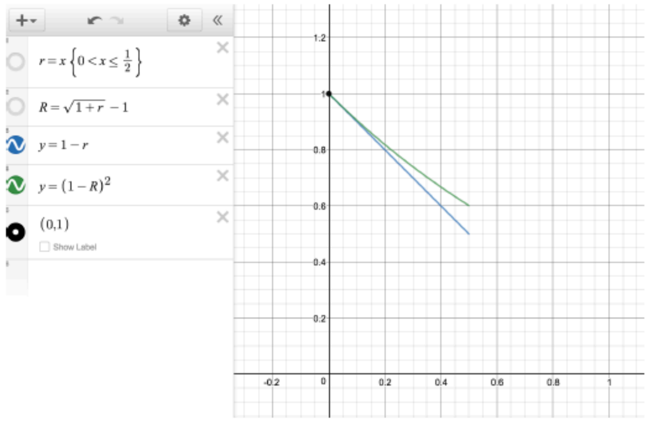

Figure 5. $1 - r < (1-R)^2$ with $R = \sqrt{1+r} - 1$.

To verify (4), first expand the condition $1+r = (1+R)^2$, from which the definition of R in (1) was obtained, subtract $4R$ from both sides, and factor to obtain $(1-R)^2 = 1+r-4R$. Substituting this in (3), yields $1-r < 1+r-4R$. By rearranging, we see we need to show that $R < \frac{1}{2}r$. Again invoking the definition of R from (15), we arrive at

$$\sqrt{1+r} < 1 + \frac{1}{2}r. \tag{5}$$

Is this true? Yes, because with $x = 1+r$, $r = x-1$, this becomes $\sqrt{x} < y$, where $y = 1 + \frac{1}{2}(x-1)$, which says that the graph of $f(x) = \sqrt(x)$ is below its tangent line at $x = 1$.

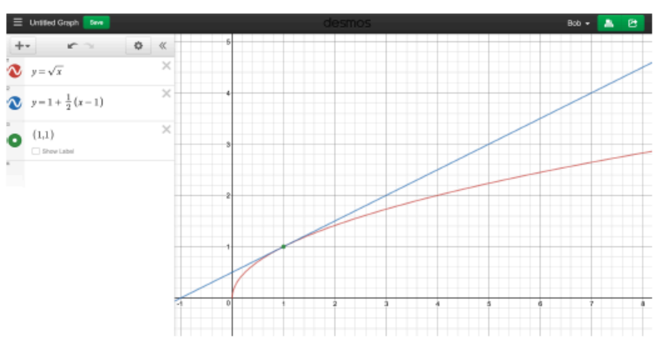

Figure 6. $f(x) = \sqrt{x} < y(x) = 1 + \frac{1}{2}(x-1)$. $\sqrt{1+r} < 1 + \frac{1}{2}r$

This is a consequence of the fact that $f(x) = \sqrt{x}$ is concave down on its domain. But since we wish to avoid arguments that depend on calculus, we try to obtain the same result by more elementary methods. Square both sides and get $1+r < (1+\frac{1}{2}r)^2 = 1+r+\frac{1}{4}r^2$ which reduces to the more elementary $0 < (\frac{r}{2})^2$. If we can reverse the steps, we are done. We obtain $1+r < (1+\frac{1}{2}r)^2$ by adding $1+r$ to both sides and factoring. Only the next and final step of applying the positive square root function, $g(x) = \sqrt{x},\ x \geq 0$ to both sides requires attention. We need to justify (without calculus) that if $u < v$ then $g(u) < g(v)$, in other words that the positive square root function is order preserving, i.e., increasing. This is a consequence of the fact that is is the inverse function of the increasing function $f(x) = x^2,\ x \geq 0$, all of which can be shown using pre-calculus arguments.

We summarize our results with a lemma.

Lemma 1: Given $r > 0$. Let $R = \sqrt{1+\min\{r, \frac{1}{2}\}} - 1$. If $|a-1| < R$ and $|b-1| < R$, then $|ab-1| < r$.

This lemma can be used to prove the analogue of Theorem 1' for ordinary sequence multiplication instead of convolution.

Theorem 1': If $a_n \to 1$ and $b_n \to 1$, then $a_n b_n \to 1$. All we need to ensure that $a_n b_n$ remains within r of 1 for $n \geq N$ is to identify the N guaranteed to exist by the hypotheses for which under the same restriction on n, both a_n and b_n are within the R from the lemma of 1. In the context of functions, if $f(x) \to 1$ and $g(x) \to 1$ as $x \to c$, we can ensure that $f(x)g(x)$ remains within r or 1 for $|x-c| < \delta$ when we identify the δ guaranteed to exist by hypotheses for which under the same restriction on x, both $f(x)$ and $g(x)$ are within the R from the lemma of 1.

Just as the special case Theorem 1' for sequence convolution implies the general case, Theorem 1, Theorem 1' implies a corresponding Theorem 2 for sequence multiplication.

Theorem 1': If $a_n \to A$ and $b_n \to B$, then $a_n b_n \to AB$.

Horizontal and vertical scaling scales both the center and dimensions of the initial rectangle $|a-1| < R,\ |b-1| < R$ to $|a-A| < AR,\ |b-B| < BR$, and simultaneously scales the bilinear bounds $|ab-1| < r$ to $|ab-AB| < ABr$. To retain the form $|ab-AB| < r$, we rescale r to $\frac{r}{AB}$.

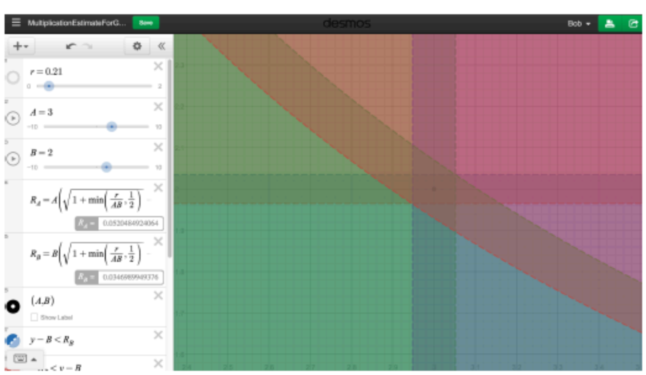

Figure 7. Horizontal scaling by A and vertical scaling by B of Figure 4 to $AB - ABr < xy < AB + ABr$.

Given $r > 0$. Let $R = \sqrt{1 + \min\{\frac{r}{|AB|}, \frac{1}{2}\}} - 1$. If $|a - A| < AR$ and $|b - B| < BR$, then $|ab - AB| < r$.

These results are sharp in the sense that for $\frac{r}{|AB|} \leq \frac{1}{2}$, one vertex of the rectangle $|a - A| \leq AR$ and $|b - B| \leq BR$ intersects one of the hyperbolas bounding the region $|ab - AB| < r$, by design. Which of the four vertices it intersects depends on the four cases of the signs of A and B.

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3. Proof of a Slightly Different Limit of a Product. Yingxian Zhu, Bob Palais, Utah Valley University

A Proof for a Slightly Different Limit of a Product

Dr. Yingxian Zhu and Dr. Bob Palais

Math Department, Utah Valley University

If $a_n \to A$, $b_n \to B$, and $c_n = \sum_{j=0}^{n} a_j b_{n-j}$ (so that $\sum_{n=0}^{\infty} c_n x^n = (\sum_{n=0}^{\infty} a_n x^n)(\sum_{n=0}^{\infty} b_n x^n)$) then $\frac{c_n}{n+1} \to AB$.

The Mean Sequence Convolution Term Approaches the Product of the Limits.

The formal power series associated with $f(x) = (1-x)^{-1} = \frac{1}{1-x}$ is the geometric series

$$1 + 1x + 1x^2 + \ldots = \sum_{n=0}^{\infty} 1x^n$$

that represents the function $f(x)$ for $x \in (-1, 1)$. We may find the derivative of both sides, in different ways. On the left, we use the power and chain rules; on the right, term-by-term differentiation of an 'infinite polynomial',

$$\frac{d}{dx}(1-x)^{-1} = -1(1-x)^{-2}(-1) = \frac{d}{dx}(1 + 1x + 1x^2 + \ldots)$$

to obtain

$$(1-x)^{-2} = (\frac{1}{1-x})^2 = 1 + 2x + 3x^2 + \ldots$$

whose interval of convergence is also $(-1, 1)$. We may also obtain the coefficients on the right using 'infinite polynomial multiplication' term-by-term, distributing, then combining terms of like degree:

$$\begin{aligned}
&(1 + x + x^2 + \ldots)(1 + x + x^2 + \ldots) \\
&= 1(1 + x + x^2 + \ldots) + x(1 + x + x^2 + \ldots) + x^2(1 + x + x^2 + \ldots) + \ldots \\
&= (1 + x + x^2 + \ldots) + (x + x^2 + \ldots) + (x^2 + \ldots) + \ldots = 1 + 2x + 3x^2 + \ldots
\end{aligned}$$

Just like with ordinary (finite) polynomial multiplication,

$$\begin{aligned}
&c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots = (a_0 + a_1 x + a_2 x^2 + \ldots)(b_0 + b_1 x + b_2 x^2 + \ldots) \\
&= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \ldots + (\sum_{j=0}^{n} a_{n-j} b_j)x^n + \ldots
\end{aligned}$$

the coefficient of the x^n term,

$$c_n = \sum_{j=0}^{n} a_{n-j} b_j \tag{1}$$

in the formal product is the sum of $n+1$ products of coefficients of the factors, $a_{n-j} b_j$, corresponding to degrees whose sum is equal to n. A product sum of this form is called a 'convolution' of the coefficients a_j with the coefficients b_j. (All subscripts refer to non-negative integers.)

For our product of geometric series example,

$$(1 + 1x + 1x^2 + \dots)(1 + 1x + 1x^2 + \dots) = 1 + 2x + 3x^2 + \dots$$

the sequences of coefficients of the factors on the left, $a_n = 1$, $b_n = 1$ for all $n \geq 0$, are constant sequences. The sequence of coefficients of the product on the right, $c_n = n+1$, becomes constant when divided by $n+1$: $\frac{c_n}{n+1} = 1$ for all $n \geq 0$. It so straightforward as to seem pointless to observe that for these constant sequences,

$$\lim_{n\to\infty} a_n = 1, \ \lim_{n\to\infty} b_n = 1, \ \lim_{n\to\infty} \frac{c_n}{n+1} = 1$$

so that

$$\lim_{n\to\infty} \frac{c_n}{n+1} = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n). \tag{2}$$

What may not be so obvious is that this equation remains valid for any two sequences $\{a_n\}.\{b_n\}$, $n \geq 0$ and $\{c_n\}$ is defined by (1), whenever the limits on the right of (2) exist. We call $\frac{c_n}{n+1}$ the 'mean convolution term' because there are $n+1$ terms in the sum for c_n. Equation (2) says that the mean convolution term approaches the product of the limits of the sequences being convolved (if their limits exist). Proving this result also provides an opportunity for another look at the role of explicit estimation in calculus.

Convolution is just a fancy name for what you've always done to find a particular coefficient of a product polynomial. (Polynomials may be considered finite power series, i.e., those for which only a finite number of coefficients are nonzero). If you think about it, this is also the basis of the algorithm we learn in elementary school to multiply two multi-digit integers, e.g., $(a_2a_1a_0)(b_2b_1b_0) = (a_2(10)^2 + a_1(10) + a_0)(b_2(10)^2 + b_1(10) + b_0)$. With all digits equal to 1, we can even see this in the pattern $(11)(11) = (\dots 21)$, $(111)(111) = (\dots 321)$, $(1111)(1111) = (\dots 4321)$, $(11111)(11111) = (\dots 54321)$

Convolution is denoted with an asterisk, so we would write $c_n = (a*b)_n$, where a identifies the sequence whose elements are a_n, in the same way the letter f identifies a function whose values are denoted $f(x)$, and a letter v can identify a vector whose components are v_n. Our sequence of convolution means is defined as

$$m_n = \frac{(a*b)_n}{n+1} = \frac{1}{n+1}\sum_{j=0}^{n} a_j b_{n-j}. \tag{3}$$

Using this notation, we can restate our result in the following theorem.

Theorem 1: Let

$$\lim_{n\to\infty} a_n = A \text{ and } \lim_{n\to\infty} b_n = B \tag{4}$$

be sequences of real numbers that converge to real numbers A and B, respectively, and denote the product of A and B as $P = AB$. Let $\{m_n\}$, $n = 0, 1, \ldots$ be the the mean convolution term sequence defined by formula (3). Then

$$\lim_{n\to\infty} m_n = P.$$

In other words,

$$\lim_{n\to\infty} \frac{(a*b)_n}{n+1} = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n). \tag{5}$$

For the same reason that convolution arise in ordinary decimal and polynomial multiplication, convolution also arises in the study of trigonometric polynomials and Fourier series, where the Fast Fourier coefficient transform is used to convert computationally expensive convolution into efficient pointwise multiplication. This is actually used in computers to make it possible to multiply two numbers with billions of digits in reasonable time! Also for this reason, estimates of the norms of convolution operations play a key role in the field known as 'harmonic analysis', for the connection between frequencies that are whole multiples of one another and musical harmony. For example:
"Theorem 3.2 shows that convolution by an L^1 function defines a bounded linear map from L^1 to L^1. In fact, convolution makes L^1 into what is called a Banach algebra - a Banach space having a bilinear and associative multiplication (denoted by $*$, say) such that the norm satisfies $||f * g|| \leq ||f||\ ||g||$" [1].

Now that we have provided some motivations for why our result may be interesting and useful, perhaps surprising but also plausible, let us proceed to the proof.

Remark: If $a_n = A,\ b_n = B,\ n = 0, 1, \ldots$ are both constant sequences, the same straightforward direct argument immediately shows that $m_n = AB,\ n = 0, 1, \ldots$ is also constant with limit AB. We can also see this without explicitly calculating convolutions by using the fact that the sequence $\{m_n\}$ depends on the sequences $\{a_n\}$ and $\{b_n\}$ in a bilinear manner. In other words, if r and s are constants that multiply two sequences a and b, respectively, then from the definition (2), the resulting mean convolution terms are multiplied by the product rs:

$$\frac{(ra*sb)_n}{n+1} = \frac{1}{n+1}\sum_{j=0}^{n}(ra_j)(sb_{n-j}) = (rs)\frac{1}{n+1}\sum_{j=0}^{n} a_j b_{n-j} = (rs)\frac{(a*b)_n}{n+1}.$$

We will use this observation to simplify the strategy and technicalities of the proof of Theorem 1, by showing it is equivalent to a special case similar to our example.

Theorem 1': Let

$$\lim_{n\to\infty} a_n = 1 \text{ and } \lim_{n\to\infty} b_n = 1 \tag{4'}$$

Then

$$\lim_{n\to\infty} \frac{(a*b)_n}{n+1} = 1. \tag{5'}$$

If we can prove Theorem 1, then Theorem 1' is an immediate corollary, since it is a particular case in which $a = b = 1$ so that $ab = 1$. But conversely, we will now show that if we can prove the special case Theorem 1', then the more general Theorem 1 is a corollary. In other words, the two formulation are actually equivalent.

Lemma 1: Theorem 1' implies Theorem 1.

Proof. Let $\{a_n\}$ and $\{b_n\}$ satisfy the conditions of Theorem 1. Define the modified (normalized!) sequences $\{aa_n = \frac{a_n}{A}\}$ and $\{bb_n = \frac{b_n}{B}\}$ so that by definition, $(aa * bb)_n = \frac{1}{AB}(a * b)_n$. The modified sequences satisfy the conditions of Theorem 1', so according to Theorem 1', $\frac{1}{AB}\frac{(a*b)_n}{n+1} = \frac{(aa*bb)_n}{n+1} \to 1$ as $n \to \infty$. Multiplying the terms of a convergent sequence by a constant multiplies the limit by the same constant. So $\frac{(a*b)_n}{n+1} = (AB)\frac{1}{AB}\frac{(a*b)_n}{n+1} \to AB$ as $n \to \infty$. QED.

The conclusion of Theorem 1', that the mean convolution term $\frac{(a*b)_n}{n+1}$ approaches 1 as n approaches infinity means that we can keep $\frac{(a*b)_n}{n+1}$ within any prescribed positive tolerance $\epsilon > 0$ of 1. by restricting n to be beyond a corresponding threshold index, $N = N(\epsilon)$. That is, for all $n > N(\epsilon)$,.

$$|\frac{(a*b)_n}{n+1} - 1| < \epsilon \tag{6'}$$

The latter inequality is equivalent to

$$1 - \epsilon < \frac{(a*b)_n}{n+1} < 1 + \epsilon$$

Remark: If we preferred to prove Theorem 1 directly, we would need to show that for the sequences in that formulation, that

$$|\frac{(a*b)_n}{n+1} - AB| < \epsilon \tag{6}$$

If we expand the convolution and index from $n = 1$ instead of $n = 0$, we obtain the interesting inequality

$$|\frac{a_1 b_n + a_2 b_{n-1} + \ldots + a_n b_1}{n} - AB| < \epsilon. \tag{7}$$

of Dr. Yingxian Zhu, that he proved for sufficiently large n under the assumptions of Theorem 1.

Our strategy for proving Theorem 1 begins by considering a convolution sum of $N+1$ terms as a sum of three contributions,

$$(a*b)_N = a_0 b_N + \ldots + a_N b_0 = C_L + C_M + C_R, \tag{8}$$

where C_L is the sum of L "left" end terms, C_R is the sum of R "right" end terms, and C_M is the sum of M "middle" terms as defined here:

$$\begin{aligned} C_L &= a_0 b_N + \ldots + a_{L-1} b_{M+R} \\ C_M &= a_L b_{M+R-1} + \ldots + a_{L+M-1} b_R \\ C_R &= a_{L+M} b_{R-1} + a_N b_0 \end{aligned} \tag{7$'$}$$

Because every term in C_M is of the form $a_j b_k$ with $j > L$ and $k > R$, we may determine L and R to exclude enough initial terms of each sequence so that both factors in the middle are sufficiently close to 1 (within ϵ_2 to guarantee that their product is within any given $\frac{\epsilon}{2}$ of 1.

1.) Because $\{a_n\} \to 1$, by definition of sequence convergence, we can find L so that if $j > L$, then $|a_j - 1| < \epsilon_2$. Because $\{b_n\} \to 1$, we can find R so that if $k > R$, then $|b_k - 1| < \epsilon_2$.

2.) We will use an estimate from which the product rule for ordinary limits follows to determine the ϵ_2 required to keep the products $a_j b_k$ within $\frac{\epsilon}{2}$ of 1. From [2],

$$\epsilon_2 = \sqrt{1 + \min\{\epsilon, \frac{1}{2}\}} - 1.$$

With this $\frac{\epsilon}{2}$, if both $|a_j - 1| < \epsilon_2$ and $|b_k - 1| < \epsilon_2$, then $|a_j b_k - 1| < \frac{\epsilon}{2}$.

Together, 1 and 2 determine a fixed number $O = L + R$, of "outer" (left and right) terms so that all middle terms are within $\frac{\epsilon}{2}$ of 1, and the mean of the middle terms must also be within $\frac{\epsilon}{2}$ of 1.

Our goal in Theorem 1 (equation (5)) is to guarantee that the convolution mean,

$$m_N = \frac{c_N}{N+1} = \frac{C_L + C_M + C_R}{N+1}, \tag{9}$$

is within ϵ of 1. Since $M + O = N + 1$, if we also define $C_O = C_L + C_R$, we may write

$$m_N = \frac{C_M + C_O}{M+O} = \frac{C_M}{M+O} + \frac{C_O}{M+O} = \frac{C_M}{M}\frac{M}{M+O} + \frac{C_O}{O}\frac{O}{M+O}. \tag{10}$$

This expresses the overall convolution mean as a weighted average of the middle mean $\overline{C_M} = \frac{C_M}{M}$ and the outer mean $\overline{C_O} = \frac{C_O}{O}$, with weights $w_M = \frac{M}{M+O}$ and $w_O = \frac{O}{M+O}$, respectively:

$$m_N = w_M \overline{C_M} + w_O \overline{C_O}. \tag{10$'$}$$

3. With the number of outer terms, O, now determined so as to guarantee

$$1-\frac{\epsilon}{2}<\overline{C_M}<1+\frac{\epsilon}{2}, \tag{11}$$

we can make the number of middle terms, M, as large as necessary so that their relative contribution to the overall mean, w_M/w_O is so great that the contribution of the outer terms cannot perturb the overall mean more than another $\frac{\epsilon}{2}$ from 1. This leaves the overall convolution mean within ϵ of 1, as required. With N defined by $N+1=L+M+R$, (5), (5′) will be satisfied.

Observe that each term of the 'outer' product sums has one factor from initial terms of a sequence, that do not depend on the number of middle terms, and a corresponding factors from the 'tail' of the other sequence, beyond the terms involved in the 'middle', which therefore depend on the number of middle terms, M. So we cannot predetermine the outer mean $\overline{C_O}$ exactly prior to choosing M But because these terms must remain at least as close to 1 as the factors in the middle, we can still bound $\overline{C_O}$. Formally, if $j>L+M$, then $j>L$, so $|a_j-1|<\epsilon_2$, and if $k>R+M$, then $k>R$, so $|b_k-1|<\epsilon_2$. Now let D_a and D_b be absolute bounds on the respective sequences, i.e.,

$$|a_n|<D_a,\ |b_n|<D_b \text{ for all } n.$$

If either bound D_a or D_b did not exist, then there must be terms in that sequence having arbitrarily large n that exceed any bound, and the sequence could not have been convergent in the first place. The magnitude the 'outer' contribution C_O is dominated by

$$D_O=L(D_a)(1+\epsilon_2)+R(D_b)(1+\epsilon_2)$$

i.e., $|C_O|<D_O$, so

$$|\overline{C_O}|<\frac{D_O}{O}=\frac{(1+\epsilon_2)(L\ D_a+R\ D_b)}{O}. \tag{12}$$

To obtain a concrete estimate for M, express the number of middle terms as a multiple, x, of the number of outer terms, i.e., $M=xO$. With this notation, the weights may be written

$$w_M=\frac{M}{M+O}=\frac{xO}{xO+O}=\frac{x}{x+1},\ w_O=\frac{O}{M+O}=\frac{O}{xO+O}=\frac{1}{x+1}.$$

Substituting these in (10′), the Nth mean convolution term is given by

$$m_N=\frac{x}{x+1}\overline{C_M}+\frac{1}{x+1}\overline{C_O}. \tag{13}$$

This makes it more clear that as x and thus M grows without bound, the contribution of $\overline{C_O}$ to m_N vanishes and m_N approaches $\overline{C_M}$. With the control on $\overline{C_M}$ from (11) and on $\overline{C_O}$ from (12), we ask the question in a slightly more general manner.

Let

$$f(x) = \frac{x}{x+1}V + \frac{1}{x+1}W, \; 0 < x < \infty. \tag{14}$$

parametrize the open segment between $W = f(0)$ and $V = f(+\infty)$. If $|V - V_o| < \frac{\epsilon}{2}$ and $|W| < D$, find an x_o so that $x > x_o$ ensures that $|f(x) - V_o| < \epsilon$. (For our purposes, $V = \overline{C_M}$, $V_o = 1$, $W = \overline{C_O}$, $D = \frac{D_O}{O}$, and then $f(x)$ gives the convolution mean m_N. Since, by the triangle inequality,

$$|f(x) - V_o| \leq |f(x) - V| + |V - V_o| < |f(x) - V| + \frac{\epsilon}{2}$$

we need to find x_o for which $x > x_o$ guarantees $|f(x) - V| < \frac{\epsilon}{2}$. For this to be true, we need

$$|f(x) - V| = |\frac{x}{x+1}V + \frac{1}{x+1}W - V| = |\frac{1}{x+1}W - \frac{1}{x+1}V| = \frac{1}{x+1}|W - V| < \frac{\epsilon}{2}.$$

To achieve this, we need

$$x > \frac{2|W - V|}{\epsilon} - 1.$$

For our purposes, V and W can depend on x, but both are restricted by bounds, $|W| < D$ and $|V - V_o| < \frac{\epsilon}{2}$. From this we know and must use that

$$|W - V| \leq |W - V_o| + |V - V_o| < |W| + |V_o| + \frac{\epsilon}{2} < D + |V_o| + \frac{\epsilon}{2}.$$

So then

$$x > \frac{2(D + |V_o| + \frac{\epsilon}{2})}{\epsilon} - 1 = \frac{2(D + |V_o|)}{\epsilon}$$

suffices to guarantee that $|f(x) - V_o| < \epsilon$.

We unravel this to express everything in terms of the parameter ϵ_2, the indices L and R beyond which terms of $\{a_n\}$, and $\{b_n\}$ remain with in ϵ_2 of 1 and the absolute bounds for those sequences, D_a and D_b. Using

$$N + 1 = L + R + M = O + M = O + xO = (1 + x)O = (1 + x)(L + R)$$

so that $N = (1 + x)(L + R) - 1$, if

$$x > \frac{2(\frac{(1+\epsilon_2)(D_a L + D_b R)}{L+R} + 1)}{\epsilon},$$

so in terms of N, if

$$N > (1 + \frac{2(\frac{(1+\epsilon_2)(D_a L + D_b R)}{L+R} + 1)}{\epsilon})(L + R) - 1, \text{ then } |m_N - 1| < \epsilon. \tag{15}$$

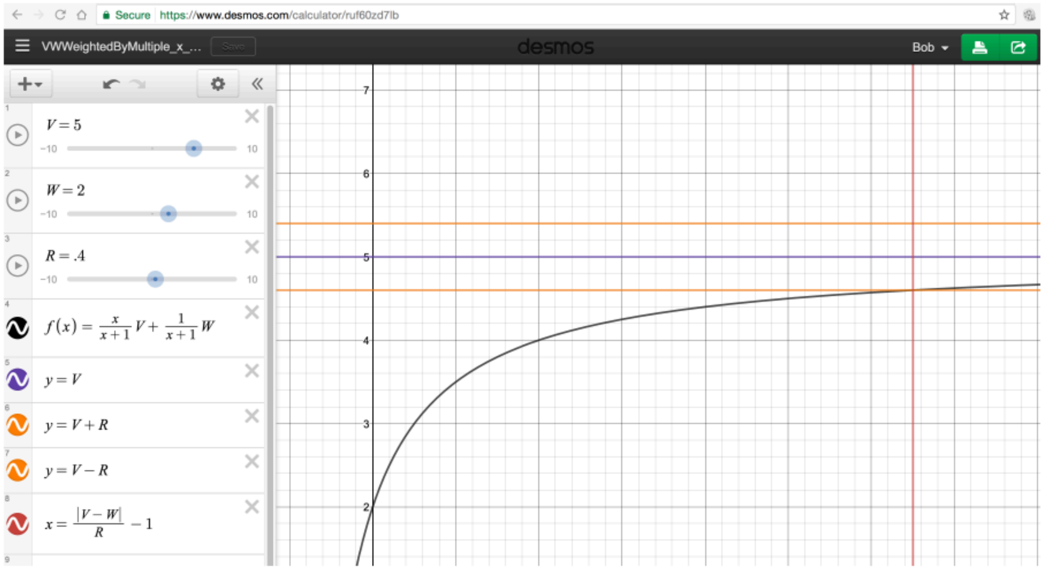

Figure 1. The behavior of a weighted average as one weight approaches 0 and the other approaches 1.

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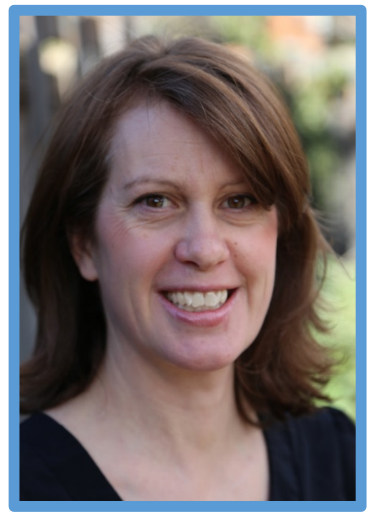
2018 UCTM Awards

Karl Jones Award-Elementary Mathematics

Tiya Karaus
Washington Elementary, Salt Lake City School District



A classroom teacher since 2000, Tiya Karaus is proud to have taught for the last eight years at Washington Elementary in the Salt Lake City School District. Tiya holds a Bachelors of Arts in history from the University of Maine and earned her teaching credentials from Loras College in Iowa.

Her passion in the classroom is guiding her students through math tasks. She has twice presented at UCTM conferences on math tasks and student discourse. Her passion outside of the classroom is spending time with her family, especially in the outdoors.

Randy Schelble Award
Special Education and Mathematics:

Carol Kaskel
Alpine School District Administration

Carol is an innovator and problem solver! Carol is the epitome of inclusion, equity and access. Carol has worked to raise expectations and have more students engage in mathematical work that is on grade level than any special education teacher I have known.

She worked with teachers and administration to create a process by which almost all students in her school were receiving grade level content with needed supports. She has done such an amazing job with mathematics that she is a teacher leader and is leading and guiding teachers across the district to move in the direction that she helped create. She recently collaborated with teachers and administration from another school in a different district to share the vision of access and equity that she has. She is a visionary when it comes to MTSS, RTI and special education.

George Shell – Secondary Mathematics

Nan Koebbe

Green Canyon High School, Cache County School District

Nan Koebbe (pronounced "Kebby") is in her 19th year of teaching secondary math. Nan returned to school at USU as a single mom with five kids at the age of 31. Upon graduation with a degree in Mathematics education and a minor in English education, she began her teaching career at Sky View High School in Smithfield, Utah, where she taught for 17 years. Last year she moved to the brand new Green Canyon High School in North Logan where she continues to enjoy teaching Secondary Math III and AP Statistics. She also taught at USU as an adjunct for the math department for several years.

Nan was instrumental in developing a bridge course at Sky View for students who struggled with Secondary math II and were not ready to go on to Math III. Although this was her greatest challenge as a teacher, it also brought her the greatest reward seeing students gain the confidence and skills they needed to continue onto higher level math courses. Nan would like to express gratitude to co-worker Connie Rawlins for her constant support and valuable insight during this experience.

Extracurricular assignments are part of daily life for any high school teacher, and Nan has served in many capacities including senior class advisor, math department chair, Academic Olympiad Coach, and graduation co-chair. She has also participated on many state committees; her favorite assignments were writing test items and curriculum guides. She is currently serving as the cheer advisor and she is part of a curriculum committee working towards bringing Standards Based Learning into practice in her school.

Nan's favorite subjects to teach are Trigonometry and Statistics, and someday she hopes to see, in addition to the crossword and Sudoku puzzle section of the newspaper, a "Verify this Trig Identity" daily puzzle. Then her life will be complete!

Nan is married to Joe Koebbe, a math professor at USU, and she has five children, 4 kids-in-law and 11 plus grandchildren, who are the most adorable and entertaining children on the planet and the delight of her life

Troy Jones

Westlake High School, Alpine School District

Troy grew up in Taylorsville, Utah. He attended the University of Utah where he long jumped on the track team and received his teaching degree in Mathematics. Troy and his wife, Cindy, moved to Richfield Utah where he accepted his first teaching position at Richfield High School, teaching for five years and also coaching the track and field and cross country teams.

Troy received his Master's Degree in Mathematics Education from BYU while at Richfield, and taught for a year at BYU as a visiting faculty member. Troy then accepted a position at the Waterford School in Sandy, Utah, where he taught for eleven years. While at Waterford, Troy purchased a plot of land in Saratoga Springs and built his own house.

When Westlake High School opened its doors in 2009, Troy transferred to Westlake in order to be closer to home and contribute to his community. He enjoys teaching math and coaching the math team at Westlake High School. Troy loves attending math conferences and workshops, where he presents and shares, along with getting great ideas from others.

He has given presentations in Spanish at math conferences in Lima Peru, Medellin Colombia, Quito Ecuador, and our very own UCTM fall conference a few years ago. Troy runs a week long summer math camp in conjunction with the Park City Mathematics Institute, where he has had 3 different Field's Medalist visit with the kids. In his spare time, Troy enjoys working on the cargo trailer he converted to a tiny home, and taking it on camping trips with the scouts and his family.

Don Clark – Lifetime Achievement

Jean Culbertson

Retired

Jean was born and attended school in Los Angeles, California. She attended the University of Redlands but missed her animals too much and transferred to USC, graduating with a math major and philosophy minor. After 3 years as actuarial clerk, she enrolled in Cal State Dominguez Hills' new teacher education program, obtaining an elementary teaching credential.

She taught at two middle schools in Manhattan Beach for 11 years. Moving to Utah was an adventure with 2 horses, 2 dogs and 5 cats. She taught USU for 3 years as she completed requirements for a secondary teaching certificate. She was hired to teach math at South Cache since several of their teachers moved to Mt. Crest. She really loved the middle school kids and spent the rest of her career at that level.

While at USU, she worked with Jim Cangelosi, who became her advisor as she worked to obtain a M.Ed. He has been a huge influence in refining her teaching methods and content selection in mathematics. They worked together to create more relevant classroom visits for pre-service mathematics teachers. As a result of this, she supervised and mentored many student teachers in South Cache. She also partnered with 2 other teachers to present in-service programs when the first NCTM standards were published and served as district math coordinator while teaching at SC.

After retiring from South Cache in 2007, she was hired full-time at USU to teach the math content courses for Elementary Education majors. That course, Math 2020, turned out to be a challenge: to cover all the content in a single semester. She and Jim finally got the OK to split the content into two semesters. She developed the Math 2010 curriculum while Jim tackled the new 2020 curriculum. She felt that she was "giving back" to the education community by teaching this course and was able to help these pre-service teachers develop a different idea about both teach and the learning of mathematics.

Now that she is really retired, she is enjoying her little hobby farm where she has 7 horses, 2 Nigerian dwarf goats, 2 dogs and 5 cats. She is honored and very humbled to receive the Don Clark Lifetime Achievement award.

2018 Elementary PAEMST Finalists

Kirk Redford

South Clearfield Elementary, Davis School District

Kirk's teaching story starts different than some other teachers. Working in the private sector for many years, he decided to start the State's Alternative Route to Licensure program. He taught fourth grade while working on his teaching certificate.

In the eleven years that he has been at South Clearfield Elementary, he has taught both 4th and 6th grade. He achieved his Master's Degree in Education from Weber State University. Has been awarded several grants to begin a 3D printing program for the 6th grade to learn STEM concepts. He has helped with Sub for Santa for his students. Is currently working on his STEM endorsement. Has begun as an instructor with the 8x8 Project, run by Davis County School District and Weber State University, which serves several school districts in Northern Utah. And he also helps facilitate a district collaborative team, where sixth grade teachers from across the district meet to share ideas and resources.

Rachel Reeder

Bridger Elementary, Logan City School District

Rachel Reeder is a first grade Spanish dual language immersion (DLI) teacher at Bridger Elementary in Logan, UT where she has taught for six years. Prior to coming to Logan she spent several years in the Wasatch County School District in Heber, UT as a third grade teacher, mathematics coach and a state DLI coordinator. Rachel received degrees from Brigham Young University (B.S. 2007) and Southern Utah University (M.A. 2009), and is a student at Utah State University (Ph.D. expected 2020).

Her degrees, certifications, endorsements, teaching experience, and current research all represent a dynamic blend of mathematics content instruction and Spanish language acquisition. Rachel enjoys traveling, gardening, serving in her church, and spending time with her family. She is especially appreciative of her husband Ryan, who is her greatest support.

2016 Secondary PAEMST Finalist

Carrie Caldwell

Hillside Middle School,
Salt Lake City School District

Carrie Caldwell has been an educator for 19 years teaching first, fourth, and fifth grades in the states of NC, KY and UT. She currently teaches seventh grade mathematics, eighth grade science, and English language development at Hillside Middle School in Salt Lake City School District. Previously, Carrie was an instructional mathematics coach for four years for SLCSD.

Carrie strives to make her mathematics classroom culturally relevant and wants all students to have access to the core curriculum regardless of academic level, gender, race, or socioeconomic background. Students discuss and explore mathematics in concrete and authentic ways to build deep understanding.

Carrie also teaches elementary mathematics methods for the University of Utah. She provides professional development to teachers on behalf of the state board of education and her district. Carrie currently serves on the Utah Council of Teachers of Mathematics board.

Carrie earned a B.A. in psychology and political science and a B.A. in elementary education from the University Kentucky. She earned a M.Ed., summa cum lade, in urban educational leadership from the University of Cincinnati. Carrie is a National Board Certified Middle-Childhood Generalist and holds elementary and level 2 mathematics endorsement.